Maximal Graphs and Spacelike Mean Curvature Flows in Semi-Euclidean Spaces

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Abstract

Two main results are proved. The first is for the maximal graph system in semi-Euclidean spaces. Existence of smooth solutions to the Dirichlet problem is proved, under certain assumptions on the boundary data. These assumptions allow the application of standard elliptic PDE methods by providing sufficiently strong a priori gradient estimates. The second result is a version of Brian White's local regularity theorem, but now for the spacelike mean curvature flow system in semi-Euclidean spaces. This is proved using a version of Huisken's monotonicity formula. Under the assumption of a suitable gradient bound, this theorem will give a priori estimates that allow such flows to be smoothly extended locally.

Declaration

The work in this thesis is based on research carried out at the Pure Mathematics Group, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction

This chapter provides a quick introduction to minimal and maximal graphs, and to mean curvature flows. To avoid wasting too much time on preliminary material, some knowledge of partial differential equations (PDEs) and semi-Riemannian manifolds is assumed. The relevant definitions and facts (for PDEs and semi-Riemannian geometry) are given in the appendix, where they will not be a distraction.

1.1 Minimal and Maximal Graphs

A *semi-Riemannian manifold* is a pair (M, g), where M is a smooth manifold and g is a metric tensor defined on M. The most obvious examples are the *semi-Euclidean spaces* $\mathbb{R}^{m+n}_n = (\mathbb{R}^{m+n}, \bar{g})$ with metric

$$\bar{g} = \sum_{i=1}^{m} dx^i \otimes dx^i - \sum_{\gamma=m+1}^{m+n} dx^\gamma \otimes dx^\gamma.$$

When $n = 1$ and $m \geq 2$, these are just the well-known *Minkowski* spaces from Relativity. When $n = 0$, they are just the Euclidean spaces $\mathbb{R}^m$ with the usual metric.

If we have a submanifold M of some semi-Euclidean space $\mathbb{R}^{m+n}_n$ then we can take the induced metric g on M in the usual way. If the induced metric is positive definite then we say that M is *spacelike*, since all tangent vectors will be spacelike in $\mathbb{R}^{m+n}_n$. Taking the Levi-Civita connections $\bar{\nabla}$ and ∇ on $\mathbb{R}^{m+n}_n$ and M (respectively), we define the *second fundamental form* by $B(V, W) = \bar{\nabla}_V W - \nabla_V W$ for any pair of tangent vector fields V, W on M. Taking the trace of B with respect to the induced metric gives a normal vector field, $H = \mathrm{trace}_g B$, on M called the *mean curvature*.

Definition 1.1.1. *An m-dimensional, spacelike submanifold of a semi-Euclidean space $\mathbb{R}^{m+n}_n$ (Euclidean space $\mathbb{R}^{m+n}$) is called a* maximal submanifold *(*minimal submanifold*) if it has mean curvature zero everywhere.*

We will consider submanifolds that can be written as graphs,

$$M = \{(x, u(x)) \mid x \in \Omega\}$$

for some smooth function $u : \Omega \to \mathbb{R}^n$ and some domain Ω in $\mathbb{R}^m$. Since $\partial/\partial x^i = (e_i, \partial u/\partial x^i)$, the induced metric on M will be given by the matrix $g(Du) = I + Du^T Du$ in $\mathbb{R}^{m+n}$ or $g(Du) = I - Du^T Du$ in $\mathbb{R}^{m+n}_n$, where I is the $m \times m$ identity matrix.[1]

Given a system of PDEs, a domain and a function defined on its boundary, a *Dirichlet problem* is the question of whether there exists a solution to the system, in this domain, with the given boundary values. For example, the problems of proving existence of minimal or maximal graphs with a given boundary. These problems are interesting for many reasons. For example, they are related to a variational problem for the volume functional, $\mathrm{Vol}[u] = \int_\Omega \sqrt{\det g(Du)}dx$. Differentiating this functional (to get its Euler system) shows that solutions of $\Delta_M u = 0$ will be critical points, where Δ_M is the induced Laplace operator on the graph of u. But $\Delta_M(x, u(x))$ gives the mean curvature vector (by Proposition A.1.1), and therefore mean curvature zero graphs are critical points. By differentiating further (to check the Legendre-Hadamard condition), we see that the minimal/maximal graph Dirichlet problem is related to the problem of minimizing/maximizing the volume among all graphs with the given boundary.[2]

[1] We consider $\mathbb{R}^m$ with the standard basis of unit vectors e_i and with coordinates denoted by x^i. We denote by $g_{ij} = \delta_{ij} \pm (\partial u^\gamma/\partial x^i)(\partial u^\gamma/\partial x^j)$ the components of g, and by g^{ij} the components of its inverse. We always use the summation convention over repeated indices $i, j, \ldots \in \{1, \ldots, m\}$ and $\gamma, \nu, \ldots \in \{m+1, \ldots, m+n\}$.

[2] See [4] or [3] for more on variational problems, and in particular for an explanation of Euler systems and the Legendre-Hadamard condition.

Problem 1.1.1. *Suppose that we are given a bounded domain Ω in $\mathbb{R}^m$ and a function $\phi : \partial\Omega \to \mathbb{R}^n$ on its boundary. Does there exist a function $u : \bar{\Omega} \to \mathbb{R}^n$, smooth in Ω and continuous on $\bar{\Omega}$, such that $u = \phi$ on $\partial\Omega$ and the graph of u over Ω is a maximal submanifold of $\mathbb{R}^{m+n}_n$ (or minimal submanifold of $\mathbb{R}^{m+n}$)?*

This is just a Dirichlet problem for the mean curvature zero system. We will see later (equation (2.1) and the comment that follows it) that the mean curvature zero system for graphs is equivalent to

$$g^{ij}(Du)\frac{\partial^2 u}{\partial x^i \partial x^j} = 0,$$

which is a second order system of PDEs. The system is elliptic since the coefficient matrix g^{ij} is positive definite, and it is quasilinear since g^{ij} depends on Du. This will be the system that we actually deal with.

In the case of minimal graphs with any dimension $m \geq 2$ and codimension $n = 1$ in $\mathbb{R}^{m+n}$, this problem involves a single equation. It is dealt with by standard elliptic methods (as seen in [10]). Jenkins and Serrin proved in [8] that if $\partial\Omega$ and ϕ are smooth, and if $\partial\Omega$ has non-negative mean curvature, then the minimal graph system has a unique smooth solution with the given domain and boundary data. Roughly, the assumption on the boundary gives an a priori gradient estimate and then higher order estimates come from the De Giorgi-Nash theorem. These allow the application of Schauder fixed point theorem (see chapter 11 of [10], or Theorem B.2.5 in the appendix here) to get existence of a solution.

Still considering the minimal graph problem, when $n > 1$ the problem becomes more difficult since the standard estimates available for single equations do not hold for systems. In [19], Lawson and Osserman give an example which shows that this problem is sometimes not solvable when $n \geq 4$, even for very 'nice' domains and boundary data. Most importantly, this tells us that it will not be possible to get a very general existence theorem as we have in the codimension 1 case.

Some examples of existence theorems for higher codimension minimal graphs can be seen in [19], [20] and [23]. Theorem 4.2 of [20] uses the inverse function theorem to prove the existence of solutions to the Dirichlet problem whenever the $C^{2,\alpha}$ norm of the boundary data is less than some constant (but this method gives us no idea of how small the constant actually is). The main result claimed in [23] is more interesting. It roughly says that a solution exists if the domain is convex and if the C^2 norm of the boundary data is bounded by some known constant. This is proved by showing that a solution to the mean curvature flow system, with boundary values given by ϕ, exists on $\Omega \times (0, \infty)$ and converges to a minimal graph.

For maximal graphs with codimension 1 in Minkowski space $\mathbb{R}^{m+1}_1$, we again only have a single equation. Therefore, if we can get an a priori gradient estimate stronger than the spacelike condition, we can again use Schauder fixed point theorem in the usual way by applying the standard higher order estimates. This problem was first dealt with by Flaherty in [6], by using boundary conditions similar to those used in the minimal graph case (as in [8]). A more general existence theorem was then proved in [2] by Bartnik and Simon (in fact, they even consider the problem of prescribed non-constant mean curvature).

For the maximal graph problem with higher codimension $n \geq 2$ in $\mathbb{R}^{m+n}_n$, very little is known. This is the case that we will consider. Unlike the higher codimension existence theorems mentioned above (in the Euclidean case), we will use standard elliptic methods by proving a suitable gradient estimate. However, we will have to deal with the fact that the higher order estimates that hold for single equations do not necessarily hold for systems. For this reason, we will only prove existence theorems either in the case of graphs with dimension $m = 2$, or for $m \geq 2$ when the gradient estimate is sufficiently strong. We can state our main results (Theorems 2.3.1 and 2.4.1) roughly as:

Claim 1.1.1. *Suppose that we are given a smooth, bounded and convex domain in $\mathbb{R}^m$, and a smooth function ϕ from the closure of this domain into $\mathbb{R}^n$. If the C^2 norm of ϕ is small enough, then there will exist a smooth maximal graph in $\mathbb{R}^{m+n}_n$, over the given domain, with boundary values given by ϕ.*

1.2 Mean Curvature Flow

Here we are interested in families of submanifolds that evolve with velocity equal to their mean curvature vector at each point, in either Euclidean or semi-Euclidean spaces. In the case of semi-Euclidean spaces, we will continue to assume that the submanifolds are spacelike.

Definition 1.2.1. *Let $\{M_t\}_{t\in I}$ be a family of smoothly embedded, spacelike, m-dimensional submanifolds M_t of $\mathbb{R}^{m+n}_n$ (or $\mathbb{R}^{m+n}$), for some interval $I \subset \mathbb{R}$. For each time $t \in I$, let $H(x,t)$ be the mean curvature vector at each point $x \in M_t$. Then this family is called a* mean curvature flow *if it satisfies the system $\partial x/\partial t = H(x,t)$.*

These flows are closely related to minimal and maximal submanifolds, which are clearly stationary solutions of this system. If each manifold in the flow is a graph over a domain Ω in $\mathbb{R}^m$,

$$M_t = \{(x, u(x,t)) \mid x \in \Omega\}$$

for some smooth $u : \Omega \times I \to \mathbb{R}^n$, then the mean curvature flow condition becomes a more useful second order, parabolic (again since g^{ij} is positive definite), quasilinear system of PDEs for u,

$$\frac{\partial u}{\partial t} = g^{ij}(Du)\frac{\partial^2 u}{\partial x^i \partial x^j}$$

where $g = I \pm Du^T Du$. This is proved later in Theorem 3.3.1.

For mean curvature flow problems, the usual goal is to prove long time existence. In other words, we want a solution on the full time interval $(0,\infty)$. For an example, see the proof of the main theorem in [23]. The idea is to split the problem up into shorter steps. The first is short time existence of a solution, on some small time interval $(0,T)$. This is usually proved by using Schauder fixed point theorem and standard methods for parabolic equations (see the proof of Theorem 8.2 of [18], or Theorem B.3.4 here). The next step, and usually the most interesting, is to extend the flow smoothly to the interval $(0,T]$. This is done by obtaining certain a priori estimates (we will explain in more detail soon). The last step is to extend the flow past time T, by applying the short time existence result again but now starting from the solution at time T. If each of these steps can be successfully completed, then they combine to tell us that the interval of existence of our solution is both open and closed in $(0, \infty)$, and therefore we will have long time existence. (For example, see [20] for a more detailed explanation of these steps.)

As mentioned above, the most interesting step is usually proving that the flow can be extended. We will therefore concentrate on this problem.

Problem 1.2.1. *Given a graphic mean curvature flow, in $\mathbb{R}^{m+n}$ or $\mathbb{R}^{m+n}_n$, which exists smoothly (and is spacelike in the semi-Euclidean case) on some time interval $(0, T)$, can we extend the flow smoothly to $(0, T]$?*

To attempt to answer this question, we need certain estimates. For example, Theorem 3.24 of [5] explains how local estimates (near some point in space) on the second fundamental form of a mean curvature flow will allow a smooth extension of the flow (in a neighbourhood of the point) to the time T. This theorem is proved for flows in Euclidean spaces, but also applies in semi-Euclidean spaces. It would also be enough to get certain Hölder estimates on the derivatives of the flow (compare to Theorem 8.3 of [18]).

In the Euclidean case, White's regularity theorem (see Theorem 3.5 of [24] or Theorem 5.6 of [5]) answers this question whenever we can prove that a quantity called the Gaussian density is close enough to 1 at time T. For spacetime points $(y, s) \in \mathbb{R}^{m+n} \times (0, T]$, the *Gaussian density* of a mean curvature flow is given by

$$\lim_{t \to s} \int_{x \in M_t} \frac{1}{(4\pi(s-t))^{m/2}} \exp\left(-\frac{|x-y|^2}{4(s-t)}\right) dx,$$

where the integrals are taken with respect to the induced metric on each M_t for times $t < s$. White's theorem says that there exists $\epsilon > 0$ such that the second fundamental form (or $C^{2,\alpha}$ norm) of the flow will be bounded in a neighbourhood of any spacetime point where the limit above is less than $1 + \epsilon$. At such a point (y, T), we then have the estimates needed to extend to time T locally near y.

Our goal will be to prove a semi-Euclidean version of White's theorem, which would allow us to deal with the problem of smoothly extending spacelike flows. Our main result (Theorem 3.6.2) will be:

Claim 1.2.1. *If we have a graphic mean curvature flow in a semi-Euclidean space, smooth on* $(0, T)$ *and satisfying a uniform gradient bound stronger than the spacelike condition, then we can extend the flow smoothly to time* T.

The idea is to define a version of the Gaussian density for spacelike flows, prove a version of White's theorem, and then use the gradient bound to show that the regularity theorem can be applied to extend the flow. We will also give an example of boundary assumptions which give the required gradient estimate.

Chapter 2

The Maximal Graph Dirichlet Problem

In this chapter, we will consider the maximal graph Dirichlet problem for higher codimension in semi-Euclidean spaces $\mathbb{R}^{m+n}_n$. We prove a gradient estimate for m-dimensional maximal graphs in $\mathbb{R}^{m+n}_n$ (where $m \geq 2$ and $n > 1$), under certain assumptions on the domain and boundary data. We use this estimate to prove existence theorems, first in the case of graphs with dimension $m = 2$, and then in the case $m \geq 2$ when the gradient estimate is strong enough.

2.1 Preliminaries

For integers $m \geq 2, n \geq 1$ we will, as usual, denote by $\mathbb{R}^{m+n}_n = (\mathbb{R}^{m+n}, \langle \cdot, \cdot \rangle)$ the semi-Euclidean space with metric tensor $\langle v, w \rangle = \sum_{i=1}^{m} v^i w^i - \sum_{\gamma=m+1}^{m+n} v^\gamma w^\gamma$. We will consider submanifolds that can be written as graphs over a domain Ω in $\mathbb{R}^m$,

$$M = \{(x, u(x)) \in \mathbb{R}^{m+n}_n \mid x \in \Omega\}$$

for some smooth $u : \Omega \to \mathbb{R}^n$. The induced metric on the graph will be given by the matrix $g = I - Du^T Du$.

It will be convenient for us to use the following norms for the maps $Du(x)$:

$\mathbb{R}^m \to \mathbb{R}^n$ and $D^2u(x) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$,

$$|||Du|||(x) = \sup_{|v|=1} |Du(x)(v)| \quad \text{and} \quad |||D^2u|||(x) = \sup_{|v|=1} |D^2u(x)(v,v)|,$$

where $|\cdot|$ denotes the usual Euclidean norm. It is possible to show that $|||Du|||^2$ will be equal to the largest eigenvalue of $Du^T Du$ at each point, and that $|||Du||| \leq |Du| \leq \sqrt{m}|||Du|||$. Using the obvious relationship between $|||Du|||$ and the eigenvalues of g, we see that the graph will be spacelike if and only if $|||Du||| < 1$, and that (for any $0 < C < 1$)[1]

$$\sqrt{\det g} \geq C \quad \Rightarrow \quad |||Du|||^2 \leq 1 - C^2,$$
$$|||Du|||^2 \leq C \quad \Rightarrow \quad \sqrt{\det g} \geq (1 - C)^{m/2}.$$

Using Proposition A.1.1, the mean curvature vector of a graph is

$$\begin{aligned} H &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^j} (x, u(x)) \right) \\ &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \right) \frac{\partial}{\partial x^j} (x, u(x)) + g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} (x, u(x)) \\ &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \right) \left(e_j, \frac{\partial u}{\partial x^j} \right) + \left(0, g^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} \right). \end{aligned} \tag{2.1}$$

Since the mean curvature vector is normal to the graph, and the first term on the right hand side is tangential, it is clear from this that the mean curvature vector is zero if and only if[2] $g^{ij}(Du)\partial^2 u/\partial x^i \partial x^j = 0$. This is a quasilinear elliptic system of n equations for u. Given a bounded domain Ω in $\mathbb{R}^m$ and boundary data $\phi : \partial\Omega \to \mathbb{R}^n$, we would therefore like to prove the existence of a smooth solution to the following Dirichlet problem:

$$g^{ij}(Du) \frac{\partial^2 u}{\partial x^i \partial x^j} = 0 \text{ and } |||Du||| < 1 \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega,$$

[1] To prove these inequalities, let λ_i^2 be the eigenvalues of $Du^T Du$ (which are all < 1 by the spacelike condition), then $1 - \lambda_i^2$ are the eigenvalues of g. If $\sqrt{\det g} \geq C$ then $\prod_i (1 - \lambda_i^2) \geq C^2$, so each $(1 - \lambda_i^2) \geq C^2$ and therefore $|||Du|||^2 \leq 1 - C^2$. If $|||Du|||^2 \leq C$ then each $\lambda_i^2 \leq C$, so $\sqrt{\det g} = \prod_i (1 - \lambda_i^2)^{1/2} \geq (1 - C)^{m/2}$.

[2] Here the 'if' direction is obvious. For the 'only if' direction, if $H = 0$ then equation (2.1) implies that the vector $v = (0, g^{ij}\partial^2 u/\partial x^i \partial x^j)$ is a tangent vector and hence can be written as $v = v^k(e_k, \partial u/\partial x^k)$, which obviously implies that each $v^k = 0$ and hence $v = 0$.

where u is at least C^2 in Ω and C^0 on $\bar{\Omega}$. Of course, we will only be able to do this under certain assumptions on the domain and boundary data. In particular, we will look for solutions in Hölder spaces. For $\alpha \in (0,1)$ we say that a function $u : \Omega \to \mathbb{R}^n$ lies in the Hölder space $C^{k,\alpha}(\bar{\Omega};\mathbb{R}^n)$ if and only if it is in $C^k(\bar{\Omega};\mathbb{R}^n)$[3] and

$$||u||_{k,\alpha} = ||u||_k + \sup_{\substack{x,y\in\bar{\Omega} \\ x\neq y}} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha} = ||u||_k + [D^k u]_\alpha$$

is finite. Here $||u||_k = \sum_{i=0}^k \sup_\Omega |D^k u|$ is the usual C^k norm. Note that, with the norm $||\cdot||_{k,\alpha}$, the space $C^{k,\alpha}(\bar{\Omega};\mathbb{R}^n)$ will be a Banach space. We will sometimes just call functions in these spaces $C^{k,\alpha}$ functions. Functions that are $C^{k,\alpha}$ on compact subsets of a domain will be called *locally* $C^{k,\alpha}$ on the domain.

We will need the following fact, which uses the Leray-Schauder fixed point theorem to get existence of solutions to the Dirichlet problem under the assumption of suitable a priori estimates.

Lemma 2.1.1. *For some $\alpha \in (0,1)$, let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^m$ and let $\phi \in C^{2,\alpha}(\bar{\Omega};\mathbb{R}^n)$. Suppose that there exist constants $\kappa \in (0,1)$ and $C > 0$ such that $\sup_\Omega |||D\phi|||^2 \leq 1-\kappa$, and such that estimates*

$$\sup_\Omega |||Du|||^2 < 1-\kappa \quad \textit{and} \quad ||u||_{1,\alpha} \leq C$$

hold whenever $u \in C^{2,\alpha}(\bar{\Omega};\mathbb{R}^n)$ satisfies $\sup_\Omega |||Du|||^2 \leq 1-\kappa$ and is a solution of the maximal graph Dirichlet problem with $u|_{\partial\Omega} = \sigma\phi|_{\partial\Omega}$ for some $\sigma \in [0,1]$. Then there exists $u \in C^{2,\alpha}(\bar{\Omega};\mathbb{R}^n)$ which is a solution to the maximal graph Dirichlet problem in $\mathbb{R}^{m+n}_n$ with boundary values $u|_{\partial\Omega} = \phi|_{\partial\Omega}$.

Most of the details can be seen in Theorem 11.4 of [10]. The proof is slightly more complicated here since $g^{ij}(Dw)$ is only positive definite when the graph of w is spacelike. This is why we need to define the set R, to avoid the non-spacelike functions for which the map T would not make sense. Also unlike Theorem 11.4 of [10], we are considering a system here. So, when we apply the Schauder estimates

[3] $C^k(\bar{\Omega};\mathbb{R}^n)$ denotes the set of functions $u : \bar{\Omega} \to \mathbb{R}^n$ with components $u^\gamma \in C^k(\bar{\Omega})$.

in this proof, we think of $g^{ij}(Dw)\partial^2 u/\partial x^i \partial x^j = 0$ as a system of decoupled linear equations (with Dw fixed in the coefficients) for the components u^γ of u. The Schauder estimates then give $C^{2,\alpha}$ bounds on each u^γ, which combine to give a $C^{2,\alpha}$ bound on u.

Proof. We let $R = \{u \in C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^n) \mid |||Du|||^2 \leq 1-\kappa\}$, and we define maps $f : C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^n) \to R$ and $T : R \to C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$.

$$f(v) = \begin{cases} v & \text{if } \sup_\Omega |||Dv|||^2 \leq 1-\kappa \\ (1-\kappa)^{1/2} v / \sup_\Omega |||Dv||| & \text{if } \sup_\Omega |||Dv|||^2 > 1-\kappa. \end{cases}$$

For any $w \in R$, we define $T(w)$ to be the unique solution u to the system of n linear Dirichlet problems given by

$$g^{ij}(Dw)\frac{\partial^2 u}{\partial x^i \partial x^j} = 0 \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega.$$

Since $w \in R$ implies that the system is elliptic and that the coefficients $g^{ij}(Du)$ are $C^{0,\alpha}$ functions,[4] we know that such a solution must exist in $C^{2,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ by the existence theorem for linear equations (see Theorem B.2.3 here, or 6.14 of [10]).

We claim that $\tilde{T} = T \circ f : C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^n) \to C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ will be continuous and compact (i.e. the images of bounded sets are precompact). The map f is continuous[5] and clearly maps bounded sets to bounded sets (with respect to the $C^{1,\alpha}$ norm). By the Schauder estimates (see Theorem B.2.2 here, or 6.6 of [10]), sets in R with bounded $C^{1,\alpha}$ norm are mapped by T to sets with bounded $C^{2,\alpha}$ norm.[6] But, by the Arzela-Ascoli theorem,[7] bounded sets in $C^{2,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ are precompact (have compact closure) in $C^2(\bar{\Omega}; \mathbb{R}^n)$ and $C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$.

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[4] $p \mapsto g^{ij}(p)$ is smooth and hence Lipschitz on $\{|||p|||^2 \leq 1-\kappa\}$, so $g^{ij}(Dw)$ is $C^{0,\alpha}$ if w is $C^{1,\alpha}$.

[5] When $v_J \to v$ in $C^{1,\alpha}$ as $J \to \infty$, obviously $\sup_\Omega |||Dv_J||| \to \sup_\Omega |||Dv|||$.

[6] The bound on the image depends only on $||\phi||_{2,\alpha}$ and the $C^{1,\alpha}$ bound on the subset of R. This is because $|u^\gamma| = |T(w)^\gamma| \leq \sup|\phi^\gamma|$ by the maximum principle, and eigenvalues of g^{ij} are ≥ 1.

[7] This says that any uniformly bounded and uniformly equicontinuous sequence of functions on $\bar{\Omega}$ has a uniformly convergent subsequence. A uniform $C^{2,\alpha}$ bound gives uniform bounds and equicontinuity on a sequence, and on the sequences of all first and second order derivatives. Arzela-Ascoli then gives a subsequence for which all derivatives up to second order converge uniformly.

Now we just need continuity of T, which we will prove exactly as in the proof of Theorem 11.4 of [10]. Let v_J be a sequence in R converging to v with respect to the $C^{1,\alpha}$ norm as $J \to \infty$. The sequence obviously must be bounded with respect to the $C^{1,\alpha}$ norm, so the set $\{T(v_J)\}$ is precompact in $C^2(\Omega; \mathbb{R}^n)$, and therefore any subsequence has a convergent subsequence. Let $T(\tilde{v}_J)$ be such a convergent subsequence, converging to some w in $C^2(\Omega; \mathbb{R}^n)$. Then by definition of T we have $0 = g^{ij}(D\tilde{v}_J)\partial^2(T(\tilde{v}_J))/\partial x^i \partial x^j$, where $g^{ij}(D\tilde{v}_J)\partial^2(T(\tilde{v}_J))/\partial x^i \partial x^j \to g^{ij}(Dv)\partial^2 w/\partial x^i \partial x^j$ by the C^2 convergence. So $g^{ij}(Dv)\partial^2 w/\partial x^i \partial x^j = 0$, and the only possible limit is $w = T(v)$. Therefore the sequence $T(v_J)$ must converge to $T(v)$.

Now we need to make use of the estimates that we have assumed to exist. Suppose that, for $\sigma \in [0,1]$, we have a fixed point $v \in C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ of the map $\sigma\tilde{T}$. We have two possible cases. First, if $\sup_\Omega |||Dv|||^2 > 1 - \kappa$ then we have $\sigma T(v\sqrt{1-\kappa}/\sup_\Omega |||Dv|||) = v$, so $w = v\sqrt{1-\kappa}/\sup_\Omega |||Dv|||$ solves the maximal graph Dirichlet problem with $w = (\sigma\sqrt{1-\kappa}/\sup_\Omega |||Dv|||)\phi$ on the boundary. But $(\sigma\sqrt{1-\kappa}/\sup_\Omega |||Dv|||) \in [0,1]$, so the assumptions that we make here imply that $\sup_\Omega |||Dw|||^2 < 1-\kappa$, which contradicts the fact that $\sup_\Omega |||Dw|||^2 = 1-\kappa$. Therefore we only need to consider the case $\sup_\Omega |||Dv|||^2 \leq 1-\kappa$, where v will be a fixed point of σT and will be a solution of the maximal graph Dirichlet problem with boundary values $\sigma\phi$. Our assumptions now imply that $||v||_{1,\alpha} \leq C$.

We conclude that $\tilde{T}$ is a compact map from the Banach space $C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ into itself and, for any $\sigma \in [0,1]$, any fixed point v of $\sigma\tilde{T}$ satisfies $||v||_{1,\alpha} \leq C$. Then Theorem 11.3 of [10][8] tells us that $\tilde{T}$ has a fixed point. As explained above, but now just taking $\sigma = 1$, this fixed point must have $|||Du|||^2 < 1-\kappa$. So it will be a spacelike solution of the maximal graph Dirichlet problem with boundary values given by ϕ. □

[8]This is a version of the Schauder fixed point theorem. It states that if F is a continuous compact map of a Banach space B to itself, and if there exists a constant C such that $|x|_B < C$ for all x in B and $\sigma \in [0,1]$ with $x = \sigma F x$, then F has a fixed point.

It is important to note that any $C^{2,\alpha}$ solution to a maximal graph Dirichlet problem (as given by the lemma) will be smooth on $\bar{\Omega}$ if the domain and boundary data are both smooth. This is proved by induction using Theorem 6.19 of [10] (also see B.2.4 in the appendix here), which says that if u is a $C^{k,\alpha}$ solution then the coefficients $g^{ij}(Du)$ are $C^{k-1,\alpha}$ and therefore u must be $C^{k+1,\alpha}$.[9] We also note that any solution with boundary data ϕ will have $|u|$ uniformly bounded in terms of $\sup_\Omega |\phi|$. This follows directly from the elliptic maximum principle (see Theorem B.2.1).

The assumption on the gradient in Lemma 2.1.1 looks ugly, but we can quickly give some examples where it holds. One example is the codimension $n = 1$ case, where the gradient estimate from section 3 of [2] can be applied. But we are really only interested in systems, so we give a more relevant example for dimension $m = 2$ and codimension $n \geq 2$. Let Ω be a bounded domain in $\mathbb{R}^2$ and let $\phi : \partial\Omega \to \mathbb{R}^n$, but now assume also that each component ϕ^γ satisfies a bounded slope condition[10] on $\partial\Omega$ with constant K_γ such that

$$\sum_\gamma K_\gamma^2 < 1 - \kappa,$$

for some $\kappa \in (0, 1)$. Then Lemma 12.6 of [10] says that, since ϕ^γ satisfies a bounded slope condition with constant K_γ, we will have $\sup_\Omega |Du^\gamma| \leq K_\gamma$ whenever u^γ is a solution of some linear elliptic equation in Ω with $u^\gamma = \phi^\gamma$ on the boundary (which will be true for any solution to the maximal graph Dirichlet problem). Combining these estimates[11] gives us $\sup_\Omega |||Du|||^2 < 1 - \kappa$. Obviously the same estimate will hold for all spacelike maximal graphs over Ω with boundary values $\sigma\phi$ for $\sigma \in [0, 1]$, thus providing a gradient estimate that we could use to apply Lemma 2.1.1.

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[9]Instead of 6.19, we could have used Theorem 6.17 of [10] which says that a *locally* $C^{k,\alpha}$ solution will be locally $C^{k+1,\alpha}$. This can be used even when the domain and data are not smooth, but only gives smoothness of solutions on the interior. Almost everything we do here could be repeated, with very little extra work, using 6.17 with non-smooth domain/data.

[10]We define the curve $\Gamma = \{(z, \phi(z)) \mid z \in \partial\Omega\}$ and we say that ϕ satisfies a *bounded slope condition with constant* K if, $\forall P \in \Gamma$, there exist planes $z \mapsto (z, \pi_P^\pm(z))$ through P such that $\pi_P^-(z) \leq \phi(z) \leq \pi_P^+(z)$ and $|D\pi_P^\pm| \leq K$ for all $z \in \partial\Omega$. (See section 12.4 of [10], or [11].)

[11]$|||Du|||^2 \leq |Du|^2 = \sum_\gamma |Du^\gamma|^2 \leq \sum_\gamma K_\gamma^2 < 1 - \kappa$.

We will prove a more useful gradient estimate in the next section for any dimension $m \geq 2$ and any codimension $n \geq 1$.

2.2 A Gradient Estimate

In this section, we will prove a gradient estimate for maximal graphs, using methods from [23] (also see chapter 14 of [10]). Although we follow [23], there are a few differences that should be pointed out. First, we are using an elliptic system, while the gradient estimate in [23] is for a parabolic system. Therefore the assumption of a gradient bound of the form $|||Du|||^2 \leq 1-\kappa$ here seems strange, but it actually does make sense given the form of Lemma 2.1.1. Secondly, the inequalities that we get for maximal graphs are slightly different to the corresponding inequalities for minimal graphs. In particular, the constant κ will appear in our inequalities in a different way, affecting the estimates that we get and the assumptions that we need to make. We therefore need to explain the details of each step here, to make sure that we get the correct inequalities, even though the structure of the proof is the same as in [23].

Given some $\kappa \in (0,1)$, we will find conditions on the bounded domain $\Omega \subset \mathbb{R}^m$ and boundary data $\phi : \bar{\Omega} \to \mathbb{R}^n$ such that any smooth solution to the corresponding maximal graph Dirichlet problem with $\sup_\Omega |||Du|||^2 \leq 1-\kappa$ must satisfy $\sup_\Omega |||Du|||^2 < 1-\kappa$. First we assume that Ω and ϕ are both C^2, and that Ω is convex.

Given such a solution u, we define a linear elliptic operator

$$L = g^{ij}(Du)\frac{\partial^2}{\partial x^i \partial x^j}.$$

Fixing any $\gamma \in \{m+1, \ldots, m+n\}$ and any $p \in \partial\Omega$, we define a function $S : \bar{\Omega} \to \mathbb{R}$ by

$$S = \nu \log(1+\zeta d) - (u^\gamma - \phi^\gamma),$$

where $d(x)$ is the distance from any point $x \in \bar{\Omega}$ to the $(m-1)$-dimensional hyperplane tangent to $\partial\Omega$ in $\mathbb{R}^m$ at the boundary point p.[12] The positive constants ν and ζ will be chosen later.

$$\frac{\partial^2}{\partial x^i \partial x^j}\nu\log(1-\zeta d) = \frac{\nu\zeta}{1+\zeta d}\frac{\partial^2 d}{\partial x^i \partial x^j} - \frac{\nu\zeta^2}{(1+\zeta d)^2}\frac{\partial d}{\partial x^i}\frac{\partial d}{\partial x^j}$$

and we already know that $Lu = 0$, and $Ld = 0$ since d is linear, so

$$LS = \frac{-\nu\zeta^2}{(1+\zeta d)^2}g^{ij}\frac{\partial d}{\partial x^i}\frac{\partial d}{\partial x^j} + L\phi^\gamma.$$

The assumed bound on the gradient tells us that the eigenvalues of g^{-1} are bounded between 1 and $1/\kappa$.[13] It is also easy to calculate that $|Dd| = 1$. These two facts, along with $d(x) \leq |x-p| \leq \text{diam}\Omega$, give

$$\frac{\nu\zeta^2}{(1+\zeta d)^2}g^{ij}\frac{\partial d}{\partial x^i}\frac{\partial d}{\partial x^j} \geq \frac{\nu\zeta^2}{(1+\zeta\text{diam}\Omega)^2}.$$

By our choice of norm for $D^2\phi$,

$$|L\phi^\gamma| \leq \frac{m}{\kappa}|||D^2\phi|||.$$

If we assume

$$\frac{\nu\zeta^2}{(1+\zeta\text{diam}\Omega)^2}\kappa \geq m|||D^2\phi|||, \tag{2.2}$$

then $LS \leq 0$, so we can apply the elliptic maximum principle to see that the infimum of S occurs on the boundary. But it is clear that $S \geq 0$ on the boundary (by the Dirichlet condition), so we have $S \geq 0$ on all of Ω and therefore $\nu\log(1+\zeta d) \geq u^\gamma - \phi^\gamma$. If we also define $S' = \nu\log(1+\zeta d) + (u^\gamma - \phi^\gamma)$, then we can repeat this to get $-\nu\log(1+\zeta d) \leq u^\gamma - \phi^\gamma$. This means that, at the point p, the normal derivative of u^γ satisfies

$$\begin{aligned}
\left|\frac{\partial(u^\gamma - \phi^\gamma)}{\partial n}\right| &= |(Du^\gamma - D\phi^\gamma)\cdot n| \\
&= \lim_{t\to 0}\frac{|[u^\gamma(p+tn) - \phi^\gamma(p+tn)] - [u^\gamma(p) - \phi^\gamma(p)]|}{t} \\
&= \lim_{x\to p}\frac{|u^\gamma(x) - \phi^\gamma(x)|}{|p-x|} \\
&\leq \lim_{d(x)\to 0}\frac{\nu\log(1+\zeta d(x))}{d(x)} = \zeta\nu,
\end{aligned}$$

[12] If we denote by n a unit normal to $\partial\Omega$ at p, then we have $d(x) = |n\cdot(x-p)|$.

[13] Since $|||Du|||^2 \leq 1-\kappa$, the eigenvalues of $g = I - Du^T Du$ are between κ and 1.

where we have used the substitution $x = p + tn$ and the fact that $\phi(p) = u(p)$. We can assume that $\partial u^\gamma/\partial n = 0$ at p for all γ except $\gamma = m+1$. We do this by rotating the coordinates of $\mathbb{R}^n$.[14] Then we have $|\partial u/\partial n| < \zeta\nu + |\partial\phi/\partial n|$. We define, at $p \in \partial\Omega$, $|||D^{\partial\Omega}u|||(p) = \sup_v |||Du(p)v|||$ where we take the supremum over unit vectors tangent to the boundary of Ω at p. Since $u = \phi$ on the boundary, $|||D^{\partial\Omega}u|||(p) = |||D^{\partial\Omega}\phi|||(p)$, so we get

$$|||Du||| \leq \left|\frac{\partial u}{\partial n}\right| + |||Du^{\partial\Omega}||| \leq \nu\zeta + \left|\frac{\partial\phi}{\partial n}\right| + |||D^{\partial\Omega}\phi||| \leq \nu\zeta + 2|||D\phi|||$$

at the point p (and hence at any boundary point). To minimize $\nu\zeta$ in such a way that inequality (2.2) holds, we take $\zeta = 1/\mathrm{diam}\Omega$ and $\nu\zeta = 4m\mathrm{diam}\Omega\sup_\Omega |||D^2\phi|||/\kappa$.[15] With this choice of constants, we have the boundary estimate

$$\sup_{\partial\Omega}|||Du||| \leq \frac{4m\mathrm{diam}\Omega}{\kappa}\sup_\Omega|||D^2\phi||| + 2\sup_{\partial\Omega}|||D\phi|||.$$

We can use this to get a gradient estimate on the full domain.

Proposition 2.2.1. *Let Ω be a bounded, convex, C^2 domain in $\mathbb{R}^m$ and let $\phi : \bar{\Omega} \to \mathbb{R}^n$ be a C^2 function. Assume, for some $\kappa \in (0,1)$, that ϕ satisfies*

$$\frac{4m\mathrm{diam}\Omega}{\kappa}\sup_\Omega|||D^2\phi||| + 2\sup_{\partial\Omega}|||D\phi||| < \sqrt{1-\kappa^{1/m}}. \tag{2.3}$$

If u is a smooth solution of the corresponding maximal graph Dirichlet problem in $\mathbb{R}^{m+n}_n$, and if $\sup_\Omega |||Du|||^2 \leq 1-\kappa$, then $\sup_\Omega |||Du|||^2 < 1-\kappa$.

Proof. To get a gradient estimate on all of $\bar{\Omega}$, we use an inequality which follows from inequality 4.6 of [16] (also see inequality (A.3) in the appendix here, taking

---

[14] By the choice of norm on $D\phi$ and $D^2\phi$ the assumptions on ϕ are preserved by this rotation, which we denote by R (since, for example, $|||D\phi|||$ involves $D\phi^T D\phi = D\phi^T R^T R D\phi$). The mean curvature zero system is also preserved (since $g = I - Du^T Du = I - Du^T R^T R Du$), so everything seen so far still holds after rotating. We can rotate back (for the same reasons) when we have the final gradient estimate.

[15] Defining $f(\nu,\zeta) = \nu\zeta$, we see that $Df = (\zeta,\nu)$, which is never zero since $\nu,\zeta > 0$. By defining $g(\nu,\zeta) = \nu\zeta^2 - (m/\kappa)\sup_\Omega |||D^2\phi|||(1+\zeta\mathrm{diam}\Omega)^2$, we use the usual Lagrange multiplier method to see that, under the condition $g = 0$, the minimum of f occurs when $\lambda Df = Dg$ for some constant λ. Solving the resulting equations for λ, ζ and ν, we get $\lambda = \zeta$ and exactly the ν and ζ given here.

the time derivative to be zero),[16]

$$\Delta_M \log \sqrt{\det g} \leq 0.$$

Applying the maximum principle to this tells us that $\sqrt{\det g} \geq \inf_{\partial\Omega} \sqrt{\det g}$ on Ω. If $|||Du|||^2 < 1 - \kappa^{1/m}$ on $\partial\Omega$ then $\sqrt{\kappa} < \inf_{\partial\Omega} \sqrt{\det g} \leq \sqrt{\det g}$. This implies that $|||Du|||^2 < 1 - \kappa$ on Ω. Now, since our boundary gradient estimate gives $|||Du|||^2 < 1-\kappa^{1/m}$ on $\partial\Omega$ whenever inequality (2.3) holds, this proves our claim. $\square$

2.3 An Existence Theorem in $\mathbb{R}^{2+n}_n$

The previous section gives us a gradient estimate as required in Lemma 2.1.1, under certain assumptions on the domain and boundary data. Now we need a suitable a priori $C^{1,\alpha}$ estimate. Since we are only interested in codimension $n > 1$, we have a system of equations, and therefore the $C^{1,\alpha}$ estimates used for single equations are not available. However, in the case of 2-dimensional graphs in $\mathbb{R}^{2+n}_n$, we can make use of the strong a priori estimates that hold for linear elliptic equations in two variables. In particular, we use the fact below, which follows directly from a comment on page 304 of [10].

Lemma 2.3.1. *Let Ω be a smooth, bounded domain in $\mathbb{R}^2$. Let $L = a^{ij}(x)\partial^2/\partial x^i \partial x^j$ be a linear elliptic operator, where $a^{ij}(x)$ is smooth on Ω with eigenvalues $\lambda(x) \leq \Lambda(x)$ such that $\Lambda/\lambda \leq \eta$ for some constant η. Let $\phi : \bar{\Omega} \to \mathbb{R}$ be smooth. Suppose that u is a C^2 solution of $Lu = 0$ in Ω, and is C^0 on $\bar{\Omega}$ with $u = \phi$ on $\partial\Omega$. Then there exist constants $\alpha, C > 0$ such that u is $C^{1,\alpha}$ on $\bar{\Omega}$ with $||u||_{1,\alpha} \leq C$. Here C depends on Ω, $||\phi||_2$ and η, while α depends on η and Ω.*[17]

This is useful to us because it applies to linear equations, rather than quasilinear equations. So we can again think of our system as a system of decoupled linear

[16]Note that the Laplace operator can be thought of as an elliptic operator on Ω by equation (A.1), which implies that $\Delta_M f = g^{ij}\partial^2 f/\partial x^i \partial x^j + (1/\sqrt{\det g})(\partial(\sqrt{\det g} g^{ij})/\partial x^j)\partial f/\partial x^i$, where g^{ij} is positive definite.

[17]If $Lu = 0$ in Ω with $u = \sigma\phi$ on the boundary, for any $\sigma \in (0, 1)$, then the linearity of L allows us to apply this lemma to u/σ to see that $||u||_{1,\alpha} \leq \sigma C$.

equations (thinking of Du as being fixed in the coefficients), and then apply the lemma to get estimates on each component of u. These combine to give an estimate on u. Unfortunately, it seems unlikely that an analogue of this lemma would exist for more than two variables (see chapter 12 of [10] for the proof and related comments).

Theorem 2.3.1. *Let Ω be a smooth, convex and bounded domain in $\mathbb{R}^2$. Let $\phi : \bar{\Omega} \to \mathbb{R}^n$ be a smooth function satisfying inequality (2.3) with $m = 2$ and $\sup_\Omega |||D\phi|||^2 \leq 1 - \kappa$ for some $\kappa \in (0, 1)$. Then there exists a smooth solution u to the maximal graph Dirichlet problem in $\mathbb{R}^{2+n}_n$ with $|||Du|||^2 < 1 - \kappa$ on $\bar{\Omega}$ and $u = \phi$ on $\partial\Omega$.*

Proof. If u is a $C^{2,\alpha}$ (for any $\alpha \in (0, 1)$) maximal graph with boundary values $\sigma\phi$ (for some $\sigma \in [0, 1]$) and $\sup_\Omega |||Du|||^2 \leq 1 - \kappa$, then the gradient estimate from the previous section gives $\sup_\Omega |||Du|||^2 < 1 - \kappa$. The eigenvalues of g^{ij} are between 1 and $1/\kappa$, so we can take $\eta = 1/\kappa$ in Lemma 2.3.1 to get an a priori $C^{1,\alpha}$ estimate on such maximal graphs (with $\alpha = \alpha(\eta, \Omega)$ as in the lemma). This allows us to apply Lemma 2.1.1 to prove the theorem. □

The result claimed in the introduction, in the case of dimension $m = 2$, now follows directly from this since the assumptions on ϕ in the above theorem will be satisfied whenever the C^2 norm of ϕ is small enough.

Corollary 2.3.1. *Given a convex, smooth, bounded domain Ω in $\mathbb{R}^2$ and any $\kappa \in (0, 1)$, there will exist a smooth solution $u : \bar{\Omega} \to \mathbb{R}^n$ to the maximal graph Dirichlet problem in $\mathbb{R}^{2+n}_n$, satisfying $|||Du|||^2 < 1 - \kappa$ and $u|_{\partial\Omega} = \phi|_{\partial\Omega}$, for any smooth $\phi : \bar{\Omega} \to \mathbb{R}^n$ with sufficiently small C^2 norm.*

Also note that, if we use the bounded slope condition mentioned in the first section of this chapter, we can prove the following theorem.

Theorem 2.3.2. *Let Ω be a bounded, smooth domain in $\mathbb{R}^2$ and let $\phi : \bar{\Omega} \to \mathbb{R}^n$ be smooth with $|||D\phi|||^2 \leq 1 - \kappa$ on $\bar{\Omega}$ for some $\kappa \in (0, 1)$. Suppose that each ϕ^γ satisfies a bounded slope condition with constant K_γ such that $\sum_\gamma K_\gamma^2 < 1 - \kappa$. Then there exists a smooth solution u to the maximal graph Dirichlet problem in $\mathbb{R}^{2+n}_n$ with $|||Du|||^2 < 1 - \kappa$ on $\bar{\Omega}$ and $u = \phi$ on $\partial\Omega$.*

Proof. As in Theorem 2.3.1, but now using the bounded slope condition to get the gradient estimate. □

2.4 An Existence Theorem in $\mathbb{R}^{m+n}_n$

Here we will prove $C^{1,\alpha}$ estimates in the case of maximal graphs in $\mathbb{R}^{m+n}_n$, of dimension $m > 2$ and codimension $n \geq 1$, whenever we have a sufficiently strong gradient estimate. These will then be used to prove an existence theorem for the Dirichlet problem. As in the case of single quasilinear equations, we get such estimates by reducing to the problem of finding estimates for supersolutions of linear equations in divergence form. Similar methods can be seen in chapter 13 of [10], but these results do not apply to systems. Since we are dealing with a system here, we will need the assumption of an estimate on the gradient strong enough that we can ignore certain terms in our inequalities. This will be made more clear later. Obviously, if our domain is convex and our boundary data has small enough C^2 norm, then the required gradient bound will exist (by the gradient estimate proved earlier).

It is important to note here that we will aim for a fast proof, rather than being careful to get the best possible estimates or the most general results. It would be possible to follow through the proofs given here more carefully, to weaken the assumptions used (or possibly even to apply them to more general quasilinear systems, under some structure conditions). However, given the form of the main theorem that we hope to prove, this does not seem to be worth the effort here. Instead we will just aim to explain the methods used as quickly and clearly as possible.

Suppose that we have a maximal graph in $\mathbb{R}^{m+n}_n$, given by a smooth function $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ with $|||Du|||^2 \leq 1 - \kappa$ for some $\kappa \in (0, 1)$. For some $\gamma \in \{m + 1, \ldots, m + n\}$ and $r \in \{1, \ldots, m\}$ to be chosen later, we define a function

$$w = \zeta\sqrt{1-\kappa}\frac{\partial u^\gamma}{\partial x^r} + v,$$

where ζ is some constant depending on m and n to be chosen later, and where

$$v = \sum_{j,\nu} \left(\frac{\partial u^\nu}{\partial x^j}\right)^2.$$

Then, writing $g(p) = I - p^T p$ for $p = (p^\nu_k)$,

$$\begin{aligned}
\frac{\partial}{\partial x^i}\left(g^{ij}(Du)\frac{\partial w}{\partial x^j}\right) &= \frac{\partial}{\partial x^i}\left(g^{ij}(Du)\right)\frac{\partial w}{\partial x^j} + g^{ij}(Du)\frac{\partial^2 w}{\partial x^i \partial x^j} \\
&= \frac{\partial g^{ij}}{\partial p^\nu_k}(Du)\frac{\partial^2 u^\nu}{\partial x^i \partial x^k}\frac{\partial w}{\partial x^j} \\
&\quad + g^{ij}(Du)\left(\zeta\sqrt{1-\kappa}\frac{\partial^3 u^\gamma}{\partial x^r \partial x^i \partial x^j} + \frac{\partial^2 v}{\partial x^i \partial x^j}\right).
\end{aligned}$$

But

$$\frac{\partial^2 v}{\partial x^i \partial x^j} = 2\frac{\partial^2 u^\nu}{\partial x^i \partial x^k}\frac{\partial^2 u^\nu}{\partial x^j \partial x^k} + 2\frac{\partial^3 u^\nu}{\partial x^k \partial x^i \partial x^j}\frac{\partial u^\nu}{\partial x^k}$$

and

$$\begin{aligned}
g^{ij}(Du)\frac{\partial^3 u^\nu}{\partial x^h \partial x^i \partial x^j} &= \frac{\partial}{\partial x^h}\left(g^{ij}(Du)\frac{\partial^2 u^\nu}{\partial x^i \partial x^j}\right) - \frac{\partial}{\partial x^h}\left(g^{ij}(Du)\right)\frac{\partial^2 u^\nu}{\partial x^i \partial x^j} \\
&= 0 - \frac{\partial g^{ij}}{\partial p^\delta_k}(Du)\frac{\partial^2 u^\delta}{\partial x^k \partial x^h}\frac{\partial^2 u^\nu}{\partial x^i \partial x^j},
\end{aligned}$$

therefore

$$\begin{aligned}
\frac{\partial}{\partial x^i}\left(g^{ij}(Du)\frac{\partial w}{\partial x^j}\right) &= \frac{\partial g^{ij}}{\partial p^\nu_k}(Du)\frac{\partial^2 u^\nu}{\partial x^i \partial x^k}\frac{\partial w}{\partial x^j} \\
&\quad -\zeta\sqrt{1-\kappa}\frac{\partial g^{ij}}{\partial p^\delta_k}(Du)\frac{\partial^2 u^\delta}{\partial x^k \partial x^r}\frac{\partial^2 u^\gamma}{\partial x^i \partial x^j} \\
&\quad +2g^{ij}(Du)\frac{\partial^2 u^\nu}{\partial x^i \partial x^h}\frac{\partial^2 u^\nu}{\partial x^j \partial x^h} \\
&\quad -2\frac{\partial u^\nu}{\partial x^h}\frac{\partial g^{ij}}{\partial p^\delta_k}(Du)\frac{\partial^2 u^\delta}{\partial x^k \partial x^h}\frac{\partial^2 u^\nu}{\partial x^i \partial x^j}. \qquad (2.4)
\end{aligned}$$

We would like to remove the second derivatives of u in the right hand side, so that we are left with a useful inequality for the left hand side. This is where we need to use the gradient estimate. We want to show that the right hand side of the above equation is dominated by the third term whenever $1-\kappa$ is small enough. First we need to remember that the eigenvalues of g^{ij} will be between 1 and $1/\kappa$, so in this third term we have

$$2g^{ij}(Du)\frac{\partial^2 u^\nu}{\partial x^i \partial x^k}\frac{\partial^2 u^\nu}{\partial x^j \partial x^k} \geq 2\sum_{\nu,k}\sqrt{\sum_i\left(\frac{\partial^2 u^\nu}{\partial x^i \partial x^k}\right)^2}\sqrt{\sum_j\left(\frac{\partial^2 u^\nu}{\partial x^j \partial x^k}\right)^2} = 2|D^2 u|^2.$$

Also, $g(p) = I - p^T p$ implies that

$$\frac{\partial g_{ij}}{\partial p^\nu_k}(p) = 0 - \delta_{ki}p^\nu_j - \delta_{kj}p^\nu_i,$$

which, by differentiating $g_{ij}g^{jh} = \delta_{ih}$, gives

$$\frac{\partial g^{fh}}{\partial p^\nu_k}(p) = g^{kf}g^{jh}p^\nu_j + g^{kh}g^{fj}p^\nu_j.$$

By the equivalence of matrix norms,[18] the Schwarz inequality[19] and the bound on the eigenvalues of g, this then gives us the inequality

$$\left|\left(\frac{\partial g^{fh}}{\partial p^\nu_k}\right)\right| \leq 2|g^{-1}|^2|p| \leq \frac{2m^2}{\kappa^2}|p|.$$

We also know that

$$\begin{aligned}\left|\frac{\partial w}{\partial x^i}\right| &= \left|\zeta\sqrt{1-\kappa}\frac{\partial^2 u^\gamma}{\partial x^i \partial x^r} + 2\frac{\partial u^\nu}{\partial x^j}\frac{\partial^2 u^\nu}{\partial x^i \partial x^j}\right| \\ &\leq |\zeta|\sqrt{1-\kappa}|D^2u| + 2|Du|\cdot|D^2u| \\ &\leq (|\zeta| + 2\sqrt{m})\sqrt{1-\kappa}|D^2u|,\end{aligned}$$

where we have used $|Du| \leq \sqrt{m}|||Du||| \leq \sqrt{m}\sqrt{1-\kappa}$. Now we can combine all of the inequalities above and apply them to equation (2.4) to get

$$\begin{aligned}\frac{\partial}{\partial x^i}\left(g^{ij}(Du)\frac{\partial w}{\partial x^j}\right) &\geq -\left|\left(\frac{\partial g^{ij}}{\partial p^\nu_k}\right)\right|\cdot|D^2u|\cdot|Dw| - |\zeta|\sqrt{1-\kappa}\left|\left(\frac{\partial g^{ij}}{\partial p^\delta_k}\right)\right|\cdot|D^2u|^2 \\ &\quad +2|D^2u|^2 - 2\cdot|Du|\cdot\left|\left(\frac{\partial g^{ij}}{\partial p^\delta_k}\right)\right|\cdot|D^2u|^2 \\ &\geq 2|D^2u|^2 - \frac{1-\kappa}{\kappa^2}C|D^2u|^2,\end{aligned}$$

where the constant $C > 0$ depends only on m and n (since ζ does). So for $1-\kappa$ small enough (how small depending on m and n) we will have

$$\frac{\partial}{\partial x^i}\left(g^{ij}(Du)\frac{\partial w}{\partial x^j}\right) \geq 0,$$

and therefore w will be a subsolution[20] of the linear divergence form equation

$$\frac{\partial}{\partial x^i}\left(\bar{g}^{ij}(x)\frac{\partial w}{\partial x^j}\right) = 0, \tag{2.5}$$

where $\bar{g}^{ij}(x) = g^{ij}(Du(x))$.

---

[18] Given any norms $|\cdot|_1$, $|\cdot|_2$ on the space of real $m \times m$ matrices, there exist constants $C, K > 0$ such that $C|A|_1 \leq |A|_2 \leq K|A|_1$ for each A. We usually apply this when $|A|_1 = \sqrt{\sum_{ij} A^2_{ij}}$ and $|A|_2$ is the square root of the largest eigenvalue of A^TA, taking $C = 1/m$ and $K = 1$.

[19] $|v \cdot w| \leq |v|\cdot|w|$ for all v, w.

[20] A function u is a *subsolution* (*supersolution*) of an elliptic equation $Lu = 0$ if $Lu \geq 0$ (≤ 0).

Lemma 2.4.1. *Let Ω be a domain in $\mathbb{R}^m$. There exist constants $\kappa, \alpha \in (0,1)$ and $K > 0$ such that if a maximal graph in $\mathbb{R}^{m+n}_n$ is given by a smooth function $u : \Omega \to \mathbb{R}^n$ with $|||Du|||^2 \leq 1 - \kappa$, then u will satisfy*

$$[Du|_{\Omega'}]_\alpha \leq K dist(\Omega', \partial\Omega)^{-\alpha}$$

on any subdomain Ω' with closure contained in Ω. Here κ, α and K depend on m and n.

Proof. First we take $\zeta > 0$ and define $w^{\pm} = \pm\zeta\sqrt{1-\kappa}\partial u^\gamma/\partial x^r + v$. Choosing a ball $B^m_{R_0}(y)$ with closure contained in Ω, we consider $R \in (0, R_0/4)$. Taking r and γ to be such that[21]

$$\mathrm{osc}_{B^m_{4R}(y)} \frac{\partial u^\gamma}{\partial x^r} \geq \mathrm{osc}_{B^m_{4R}(y)} \frac{\partial u^\nu}{\partial x^i}$$

for all $i = 1, \ldots, m$ and $\nu = m+1, \ldots m+n$. We easily check that[22]

$$\sqrt{1-\kappa}(\zeta - 2mn)\mathrm{osc}_{B^m_{4R}(y)} \frac{\partial u^\gamma}{\partial x^r} \leq \mathrm{osc}_{B^m_{4R}(y)} w^{\pm} \leq \sqrt{1-\kappa}(\zeta + 2mn)\mathrm{osc}_{B^m_{4R}(y)} \frac{\partial u^\gamma}{\partial x^r}. \tag{2.6}$$

Choosing $\zeta = 10mn$ and setting $W^{\pm} = \sup_{B^m_{4R}(y)} w^{\pm}$, we get (using (2.6) and the footnote again)

$$\begin{aligned}
\inf_{B^m_{4R}(y)} \sum_{+,-}(W^{\pm} - w^{\pm}) &\geq \sup_{B^m_{4R}(y)} \left(\zeta\sqrt{1-\kappa}\frac{\partial u^\gamma}{\partial x^r} + \inf_{B^m_{4R}(y)} v\right) \\
&\quad + \sup_{B^m_{4R}(y)} \left(-\zeta\sqrt{1-\kappa}\frac{\partial u^\gamma}{\partial x^r} + \inf_{B^m_{4R}(y)} v\right) - 2 \sup_{B^m_{4R}(y)} v \\
&= \sqrt{1-\kappa}\zeta\left(\sup_{B^m_{4R}(y)} \frac{\partial u^\gamma}{\partial x^r} - \inf_{B^m_{4R}(y)} \frac{\partial u^\gamma}{\partial x^r}\right) - 2(\sup_{B^m_{4R}(y)} v - \inf_{B^m_{4R}(y)} v) \\
&= \sqrt{1-\kappa}\zeta \mathrm{osc}_{B^m_{4R}(y)} \frac{\partial u^\gamma}{\partial x^r} - 2\mathrm{osc}_{B^m_{4R}(y)} v \\
&\geq \sqrt{1-\kappa}(\zeta - 4mn)\mathrm{osc}_{B^m_{4R}(y)} \frac{\partial u^\gamma}{\partial x^r} \\
&\geq \frac{\zeta - 4mn}{\zeta + 2mn}\mathrm{osc}_{B^m_{4R}(y)} w^{\pm} \\
&= \frac{\mathrm{osc}_{B^m_{4R}(y)} w^{\pm}}{2}.
\end{aligned} \tag{2.7}$$

[21] $\mathrm{osc}_B f = \sup_{x,y \in B} |f(x) - f(y)|$.

[22] Here we use $v(x) - v(y) = \sum_{\nu,j}[(\partial u^\nu/\partial x^j)^2(x) - (\partial u^\nu/\partial x^j)^2(y)] = \sum_{\nu,j}[(\partial u^\nu/\partial x^j)(x) + (\partial u^\nu/\partial x^j)(y)][(\partial u^\nu/\partial x^j)(x) - (\partial u^\nu/\partial x^j)(y)] \leq \sum_{\nu,j}(2\sup|||Du|||)\mathrm{osc}(\partial u^\nu/\partial x^j) \leq 2mn(1-\kappa)^{1/2}\mathrm{osc}(\partial u^\gamma/\partial x^r)$ by the choice of γ, r.

The choice of ζ is the last real difference between the system case and the equation case for this proof, but we will give the few remaining details just so that the steps above make sense. From here we can apply the weak Harnack inequality[23] (Theorem 8.18 of [10]), to the supersolution $W^\pm - w^\pm$ of equation (2.5), to get

$$(\inf_{B_R^m(y)}(W^\pm - w^\pm) + 0) \geq \frac{C}{R^m}\int_{B_{2R}^m(y)}(W^\pm - w^\pm) \tag{2.8}$$

for some positive constant C depending on κ and m. By inequality (2.7), we have $\inf_{B_{4R}^m(y)}(W^\pm - w^\pm) \geq \mathrm{osc}_{B_{4R}^m(y)} w^\pm/4$ for either w^+ or w^-. Assume that it holds for w^+, then

$$\frac{C}{R^m}\int_{B_{2R}^m(y)}(W^+ - w^+) \geq C'\mathrm{osc}_{B_{4R}^m(y)} w^+ \tag{2.9}$$

for some constant C' depending on κ and m. If we define $\bar{\omega}_{r\gamma}(R) = \mathrm{osc}_{B_{4R}^m(y)} w^+$ and $\omega_{i\nu}(R) = \mathrm{osc}_{B_{4R}^m(y)}(\partial u^\nu/\partial x^i)$. Then inequalities (2.8) and (2.9) give

$$\begin{aligned}
C'\bar{\omega}_{r\gamma}(R) &\leq \inf_{B_R^m(y)}(W^+ - w^+)\\
&\leq \sup_{B_{4R}^m(y)} w^+ - \inf_{B_{4R}^m(y)} w^+ + \inf_{B_R^m(y)} w^+ - \sup_{B_R^m(y)} w^+\\
&= \bar{\omega}_{r\gamma}(R) - \bar{\omega}_{r\gamma}(R/4),
\end{aligned}$$

so $\bar{\omega}_{r\gamma}(R/4) \leq \bar{\omega}_{r\gamma}(R)(1-C')$. It also is easy to see that $\bar{\omega}_{r\gamma}(R_0/4) \leq \sup_{B_{R_0}^m(y)} w^+ \leq C''(1-\kappa)$ for some constant C'' depending on m and n, and by inequality (2.6) that

$$\omega_{i\nu}(R) \leq \mathrm{osc}_{B_{4R}^m(y)}\frac{\partial u^\gamma}{\partial x^r} \leq \frac{\mathrm{osc}_{B_{4R}^m(y)} w^+}{\sqrt{1-\kappa}8mn} = \frac{\bar{\omega}_{r\gamma}(R)}{\sqrt{1-\kappa}8mn}.$$

These facts allow us to apply Lemma 13.5 of [10][24] to get $\omega_{i\nu}(R) \leq C'''R^\alpha/R_0^\alpha$ for each i and ν, and for all $R \in (0, R_0/4)$, where the constant C''' depends on m, n, κ.

[23] Let L be a linear elliptic operator, $Lw = (\partial/\partial x^i)(a^{ij}(x)\partial w/\partial x^j)$ on Ω, with eigenvalues of a^{ij} between two positive constants $\lambda \leq \Lambda$. Let $f \in L^m(\Omega)$ and let $w \in C^2(\bar{\Omega})$ be a supersolution of $Lw = f$ in Ω with $w \geq 0$ in $B_{4R}^m(y) \subset \Omega$. Then $||w||_{L^1(B_{2R}^m(y))} \leq R^m C(\inf_{B_R^m(y)} w + \lambda^{-1}R||f||_{L^m(\Omega)})$, for some constant C depending on m and Λ/λ. Here $L^p(\Omega)$ is the set of functions with $||u||_{L^p(\Omega)} = (\int_\Omega |u|^p dx)^{1/p} < \infty$ (for $p \geq 1$).

[24] Let $\{\omega_A\}$ and $\{\bar{\omega}_A\}$ for $A = 1, \ldots, N$ be sequences of non-decreasing functions on an interval $(0, R_0)$, such that for each $R \leq R_0$ there exists $\bar{\omega}_B \in \{\bar{\omega}_A\}$ with $\bar{\omega}_B(R) \geq$ every $\delta_0\omega_A(R)$ (for some constant $\delta_0 > 0$) and such that $\bar{\omega}_B(R/4) \leq \gamma\bar{\omega}_B(R) + \sigma(R)$ (for some non-decreasing σ and some constant $\gamma > 0$). Then, for each $R \leq R_0$, we have $\omega_A(R) \leq C[(R/R_0)^\alpha \max_A \bar{\omega}_A(R_0) + \sigma(\sqrt{R_0 R})]$, for some constants α, C depending on N, γ, δ_0.

We finally take $2R_0$ be the distance between Ω' and $\partial\Omega$ (for example). If $x, y \in \Omega'$ with $|x-y| = R < R_0/4$ then $|(\partial u^\nu/\partial x^i)(x) - (\partial u^\nu/\partial x^i)(y)| \leq \operatorname{osc}_{B^m_{4R}(y)} \partial u^\nu/\partial x^i$, which is less than $(R/R_0)^\alpha = |x-y|^\alpha/R_0^\alpha$ multiplied by some constant depending on m, n and κ. Obviously if $|x-y| \geq R_0/4$ then, by the gradient bound, $|(\partial u^\nu/\partial x^i)(x) - (\partial u^\nu/\partial x^i)(y)|$ is less than $|x-y|^\alpha/R_0^\alpha$ multiplied by some constant depending on κ. This gives the expected estimate. $\square$

This lemma is just an interior estimate, and we really need an estimate on the full domain. Therefore we will need some kind of $C^{1,\alpha}$ estimate at the boundary of Ω. To do this, we need to adjust our problem in such a way that we only need to consider a solution which is zero on a flat boundary portion. We will need to be careful about where the gradient (of a solution) appears in our inequalities. We have to make sure that, as before, a strong enough bound on the gradient will allow us to assume that certain terms dominate. Unfortunately, the fact that we have to transform our domain and boundary data means that the gradient bound needed will depend on the transformation and therefore on the original domain Ω and boundary data ϕ.

Given a smooth, bounded domain Ω in $\mathbb{R}^m$, let B be some ball in $\mathbb{R}^m$ with centre on $\partial\Omega$. Taking B to be smaller if necessary, we can assume that there is a coordinate change $F : B \to F(B) \subset \mathbb{R}^m$ such that F and F^{-1} are smooth, with

$$F(B \cap \partial\Omega) \subset \{y \mid y^m = 0\} \quad \text{and} \quad F(B \cap \Omega) \subset \{y \mid y^m > 0\},$$

and such that the matrix $DFDF^T$ has eigenvalues bounded from above and below by positive constants Λ_F and λ_F respectively (since this matrix can be assumed to be smooth and positive definite on the closure of B).

We assume that a smooth function $u : \bar{\Omega} \to \mathbb{R}^n$ gives a maximal graph with $|||Du|||^2 \leq 1-\kappa$. We also assume a Dirichlet boundary condition, $u|_{\partial\Omega} = \phi|_{\partial\Omega}$ for some smooth $\phi : \bar{\Omega} \to \mathbb{R}^n$, where $||\phi||_2 \leq \Phi_2$ and $||\phi||_3 \leq \Phi_3$ for some positive constants Φ_2 and Φ_3. We define $\tilde{u}$ by $\tilde{u}(F(x)) = u(x)$. Then $Du = D\tilde{u}DF$ and

$$g_{ij}(Du) = \delta_{ij} - \frac{\partial u^\nu}{\partial x^i}\frac{\partial u^\nu}{\partial x^j} = \delta_{ij} - \left(\frac{\partial \tilde{u}^\nu}{\partial y^k}\frac{\partial F^k}{\partial x^i}\right)\left(\frac{\partial \tilde{u}^\nu}{\partial y^h}\frac{\partial F^h}{\partial x^j}\right).$$

If we define

$$A_{ij}(y, D\tilde{u}(y)) = \delta_{ij} - \left(\frac{\partial \tilde{u}^\nu}{\partial y^k}(y)\frac{\partial F^k}{\partial x^i}(F^{-1}(y))\right)\left(\frac{\partial \tilde{u}^\nu}{\partial y^h}(y)\frac{\partial F^h}{\partial x^j}(F^{-1}(y))\right),$$

then

$$\begin{aligned} 0 &= g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} \\ &= \left(\frac{\partial F^k}{\partial x^i}A^{ij}\frac{\partial F^h}{\partial x^j}\right)\frac{\partial^2 \tilde{u}}{\partial y^k \partial y^h} + A^{ij}\frac{\partial^2 F^k}{\partial x^i \partial x^j}\frac{\partial \tilde{u}}{\partial y^k}. \end{aligned}$$

We also define $\tilde{\phi}$ by $\phi = \tilde{\phi}(F)$, and take $\hat{u} = \tilde{u} - \tilde{\phi}$ so that $\hat{u} = 0$ at points on $F(\partial\Omega \cap B)$. Then $\hat{u}$ satisfies the system

$$\begin{aligned} 0 &= \left(\frac{\partial F^k}{\partial x^i}A^{ij}\frac{\partial F^h}{\partial x^j}\right)\frac{\partial^2 \hat{u}}{\partial y^k \partial y^h} \\ &\quad + \left(A^{ij}\frac{\partial^2 F^k}{\partial x^i \partial x^j}\right)\left(\frac{\partial \hat{u}}{\partial y^k} + \frac{\partial \tilde{\phi}}{\partial y^k}\right) + \left(\frac{\partial F^k}{\partial x^i}A^{ij}\frac{\partial F^h}{\partial x^j}\right)\frac{\partial^2 \tilde{\phi}}{\partial y^k \partial y^h}, \end{aligned}$$

where the coefficients $A^{ij} = A^{ij}(y, D\tilde{u}) = A^{ij}(y, D\hat{u} + D\tilde{\phi})$, and where the matrix $A^{-1} = (A^{ij})$ has eigenvalues between 1 and $1/\kappa$. If we define

$$\begin{aligned} G^{kh}(y, D\hat{u}) &= A^{ij}(y, D\hat{u} + D\tilde{\phi})\frac{\partial F^k}{\partial x^i}(F^{-1})\frac{\partial F^h}{\partial x^j}(F^{-1}), \\ B(y, D\hat{u}) &= A^{ij}(y, D\hat{u} + D\tilde{\phi})\frac{\partial^2 F^k}{\partial x^i \partial x^j}(F^{-1})\left(\frac{\partial \hat{u}}{\partial y^k} + \frac{\partial \tilde{\phi}}{\partial y^k}\right) + G^{kh}(y, D\hat{u})\frac{\partial^2 \tilde{\phi}}{\partial y^h \partial y^k}, \end{aligned}$$

where the matrix $G^{-1} = (G^{kh})$ has eigenvalues between λ_F and Λ_F/κ, then we have the elliptic system

$$0 = G^{kh}(y, D\hat{u}(y))\frac{\partial^2 \hat{u}}{\partial y^k \partial y^h} + B(y, D\hat{u}(y)). \tag{2.10}$$

Now we define a function

$$w = \zeta\sqrt{1-\kappa}\frac{\partial \hat{u}^\gamma}{\partial y^r} + \sum_\nu \sum_{\ell=1}^{m-1}\left(\frac{\partial \hat{u}^\nu}{\partial y^\ell}\right)^2$$

for some $\gamma \in \{m+1, \ldots, m+n\}$ and $r \in \{1, \ldots, m-1\}$. It is important to remember that we will apply the summation convention over the usual ranges for all indices except ℓ, where we will only take sums over $\ell = 1, \ldots, m-1$. This means that w

only involves the tangential derivatives at the flat boundary portion. We get

$$\begin{aligned}
\frac{\partial}{\partial y^i}\left(G^{ij}\frac{\partial w}{\partial y^j}\right) &= \frac{\partial}{\partial y^i}(G^{ij})\frac{\partial w}{\partial y^j} \\
&\quad + G^{ij}\left(\zeta\sqrt{1-\kappa}\frac{\partial^3\hat{u}^\gamma}{\partial y^i\partial y^j\partial y^r} + 2\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^\ell}\frac{\partial^2\hat{u}^\nu}{\partial y^j\partial y^\ell} + 2\frac{\partial\hat{u}^\nu}{\partial y^\ell}\frac{\partial^3\hat{u}^\nu}{\partial y^\ell\partial y^i\partial y^j}\right) \\
&= \frac{\partial}{\partial y^i}(G^{ij})\frac{\partial w}{\partial y^j} + \zeta\sqrt{1-\kappa}\frac{\partial}{\partial y^r}\left(G^{ij}\frac{\partial^2\hat{u}^\gamma}{\partial y^i\partial y^j}\right) \\
&\quad - \zeta\sqrt{1-\kappa}\frac{\partial}{\partial y^r}(G^{ij})\frac{\partial^2\hat{u}^\gamma}{\partial y^i\partial y^j} + 2G^{ij}\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^\ell}\frac{\partial^2\hat{u}^\nu}{\partial y^j\partial y^\ell} \\
&\quad + 2\frac{\partial\hat{u}^\nu}{\partial y^\ell}\frac{\partial}{\partial y^\ell}\left(G^{ij}\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^j}\right) - 2\frac{\partial\hat{u}^\nu}{\partial y^\ell}\frac{\partial}{\partial y^\ell}(G^{ij})\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^j} \\
&= \frac{\partial}{\partial y^i}(G^{ij})\frac{\partial w}{\partial y^j} - \zeta\sqrt{1-\kappa}\frac{\partial}{\partial y^r}(B^\gamma) \\
&\quad - \zeta\sqrt{1-\kappa}\frac{\partial}{\partial y^r}(G^{ij})\frac{\partial^2\hat{u}^\gamma}{\partial y^i\partial y^j} + 2G^{ij}\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^\ell}\frac{\partial^2\hat{u}^\nu}{\partial y^j\partial y^\ell} \\
&\quad - 2\frac{\partial\hat{u}^\nu}{\partial y^\ell}\frac{\partial}{\partial y^\ell}(B^\nu) - 2\frac{\partial\hat{u}^\nu}{\partial y^\ell}\frac{\partial}{\partial y^\ell}(G^{ij})\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^j}. \qquad (2.11)
\end{aligned}$$

We will use the obvious fact (where we will take $z = y^k$ or p^ν_k) that[25]

$$\frac{\partial A^{ij}}{\partial z} = -A^{ih}A^{jk}\frac{\partial A_{kh}}{\partial z} \quad \Rightarrow \quad \left|\left(\frac{\partial A^{ij}}{\partial z}\right)\right| \leq |A^{-1}|^2\left|\left(\frac{\partial A_{kh}}{\partial z}\right)\right| \leq \frac{C}{\kappa^2}\left|\left(\frac{\partial A_{kh}}{\partial z}\right)\right|,$$

for some constant C depending on m. We will also stop labelling constants here and will, for now, just denote all constants by C. First,

$$\begin{aligned}
\left|\left(\frac{\partial A_{ij}}{\partial y^k}\right)\right|(y,p) &\leq |0| + 2\left|\left(\frac{\partial}{\partial y^k}\left(\frac{\partial F^h}{\partial x^i}\right)p^\nu_h p^\nu_f\frac{\partial F^f}{\partial x^j}\right)\right| \leq C|p|^2 \\
\Rightarrow \left|\left(\frac{\partial A^{ij}}{\partial y^k}\right)\right|(y,p) &\leq |A^{-1}|^2\left|\left(\frac{\partial A_{ij}}{\partial y^k}\right)\right| \leq \frac{C}{\kappa^2}|p|^2,
\end{aligned}$$

where the constants depend only on F and m. Similarly,

$$\left|\left(\frac{\partial A^{ij}}{\partial p^\nu_k}\right)\right| \leq |A^{-1}|^2\left|\left(\frac{\partial A_{ij}}{\partial p^\nu_k}\right)\right| \leq \frac{C}{\kappa^2}\left|\left(\frac{\partial F^h}{\partial x^i}\delta_{\nu\eta}\delta_{hk}p^\nu_f\frac{\partial F^f}{\partial x^j}\right)\right| \leq \frac{C}{\kappa^2}|p|,$$

where the constants depend on F and m again. We can use these inequalities to get

$$\begin{aligned}
\left|\frac{\partial}{\partial y^k}\left(G^{ij}(y, D\hat{u}(y))\right)\right| &= \left|\frac{\partial}{\partial y^k}\left(A^{fh}(y, D\hat{u} + D\tilde{\phi})\frac{\partial F^i}{\partial x^f}\frac{\partial F^j}{\partial x^h}\right)\right| \\
&\leq \frac{C}{\kappa^2}\left(|D\hat{u} + D\tilde{\phi}|^2 + |D^2\hat{u} + D^2\tilde{\phi}||D\hat{u} + D\tilde{\phi}| + 1\right),
\end{aligned}$$

[25] Here the notation $|(\cdot)|$ indicates that we are taking the norm of a matrix, not just a component.

and similarly,

$$\begin{aligned}\left|\frac{\partial}{\partial y^k}\left(B^\nu(y, D\hat{u}(y))\right)\right| \;\leq\; & \frac{C}{\kappa^2}\Big(|D\hat{u}+D\tilde{\phi}|^3 + |D\hat{u}+D\tilde{\phi}|^2|D^2\hat{u}+D^2\tilde{\phi}|\Big)\\ & +\frac{C}{\kappa^2}\Big(|D\hat{u}+D\tilde{\phi}| + |D^2\hat{u}+D^2\tilde{\phi}|\Big)\\ & +\frac{C}{\kappa^2}\Big(|D\hat{u}+D\tilde{\phi}|^2 + |D^2\hat{u}+D^2\tilde{\phi}|+1\Big)|D^2\tilde{\phi}|\\ & +\frac{C}{\kappa}|D^3\tilde{\phi}|,\end{aligned}$$

where all constants again depend on F and m. Applying the Schwarz, triangle and Young[26] inequalities, we get

$$\begin{aligned}&|D\hat{u}+D\tilde{\phi}| = |D\tilde{u}| \leq |Du|\cdot|DF| \leq C\sqrt{1-\kappa} \leq C,\\ &|Dw| \leq C|D^2\hat{u}|\sqrt{1-\kappa},\\ &|D^2\hat{u}+D^2\tilde{\phi}| \leq |D^2\hat{u}| + |D^2\tilde{\phi}|,\\ &|D^2\hat{u}| \leq |D^2\hat{u}|^2/2 + 1/2,\end{aligned}$$

where the constants C depend on m, n and F. We apply all of the inequalities above, along with the Schwarz inequality and $0 < \kappa < 1$, to equation (2.11) to get

$$\frac{\partial}{\partial y^i}\left(G^{ij}\frac{\partial w}{\partial y^j}\right) \geq 2G^{ij}\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^\ell}\frac{\partial^2\hat{u}^\nu}{\partial y^j\partial y^\ell} - \frac{C\sqrt{1-\kappa}}{\kappa^2}\Big(|D^2\hat{u}|^2 + 1 + |D^3\tilde{\phi}|\Big) \tag{2.12}$$

for some constant C depending on m, n, F and Φ_2 (by dependence on the second derivatives of $\tilde{\phi}$). We can take the coefficient of the second term on the right hand side to be small by taking κ to be close to 1. We hope that the term

$$2G^{ij}\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^\ell}\frac{\partial^2\hat{u}^\nu}{\partial y^j\partial y^\ell} \geq 2\lambda_F\sum_\nu\sum_i\sum_{\ell=1}^{m-1}\left(\frac{\partial^2\hat{u}^\nu}{\partial y^i\partial y^\ell}\right)^2 \tag{2.13}$$

will dominate when κ is close enough to 1. Obviously this term contains all second order derivatives of $\hat{u}$ except $\partial^2\hat{u}/\partial x^m\partial x^m$. By using the system satisfied by $\hat{u}$, and the obvious bounds on $|B|$ and $|G^{-1}|$, we can estimate this remaining second

[26] $\pm ab \leq \epsilon a^2/2 + b^2/2\epsilon$ for $a, b \in \mathbb{R}$ and $\epsilon > 0$.

derivative

$$
\begin{aligned}
\left|\frac{\partial^2 \hat{u}^\nu}{\partial y^m \partial y^m}\right| &= \left|\frac{1}{G^{mm}}\left(\sum_{(i,j)\neq(m,m)} G^{ij}\frac{\partial^2 \hat{u}^\nu}{\partial y^i \partial y^j} - B^\nu\right)\right| \\
&\leq C\left(|G^{-1}|\sqrt{\sum_{(i,j)\neq(m,m)}\left(\frac{\partial^2 \hat{u}^\nu}{\partial y^i \partial y^j}\right)^2} + |B|\right) \\
&\leq \frac{C}{\kappa}\left(\sqrt{\sum_{(i,j)\neq(m,m)}\left(\frac{\partial^2 \hat{u}^\nu}{\partial y^i \partial y^j}\right)^2} + (|D\hat{u} + D\tilde{u}| + |D^2\tilde{\phi}|)\right) \\
&\leq \frac{C}{\kappa}\left(\sqrt{\sum_{(i,j)\neq(m,m)}\left(\frac{\partial^2 \hat{u}^\nu}{\partial y^i \partial y^j}\right)^2} + \sqrt{1-\kappa} + |D^2\tilde{\phi}|\right) \\
&\leq \frac{C}{\kappa}\left(\sqrt{\sum_{(i,j)\neq(m,m)}\left(\frac{\partial^2 \hat{u}^\nu}{\partial y^i \partial y^j}\right)^2} + 1\right), \qquad (2.14)
\end{aligned}
$$

where C depends on m, n, F and Φ_2 but not κ. This implies that

$$
\begin{aligned}
|D^2\hat{u}|^2 &= \sum_\nu\left(\sum_{(i,j)\neq(m,m)}\left(\frac{\partial^2 \hat{u}^\nu}{\partial y^i \partial y^j}\right)^2 + \left(\frac{\partial^2 \hat{u}^\nu}{\partial y^m \partial y^m}\right)^2\right) \\
&\leq \frac{C}{\kappa^2}\left(\sum_\nu \sum_{(i,j)\neq(m,m)}\left(\frac{\partial^2 \hat{u}^\nu}{\partial y^i \partial y^j}\right)^2 + 1\right), \qquad (2.15)
\end{aligned}
$$

where we have used Young's inequality and $\kappa < 1$, and where C depends on m, n, F and Φ_2. The right hand side of this inequality again contains all derivatives except $\partial^2\hat{u}/\partial x^m \partial x^m$. Combining this with (2.13) gives

$$
2G^{ij}\frac{\partial^2 \hat{u}^\nu}{\partial y^i \partial y^\ell}\frac{\partial^2 \hat{u}^\nu}{\partial y^j \partial y^\ell} \geq C(\kappa^2|D^2\hat{u}|^2 - 1),
$$

and then inequality (2.12) gives (for constants again depending on m, n, F, Φ_2)

$$
\frac{\partial}{\partial y^i}\left(G^{ij}\frac{\partial w}{\partial y^j}\right) \geq C\left(\kappa^2|D^2\hat{u}|^2 - 1\right) - \frac{C\sqrt{1-\kappa}}{\kappa^2}\left(|D^2\hat{u}|^2 + 1 + |D^3\tilde{\phi}|\right),
$$

which implies that if we choose κ close enough to 1 then the terms involving $|D^2\hat{u}|^2$ will cancel. This leaves

$$
\frac{\partial}{\partial y^i}\left(G^{ij}\frac{\partial w}{\partial y^j}\right) \geq C,
$$

for some (not necessarily positive) constant C depending on m, n, F and Φ_3. Therefore w will be a subsolution of a linear divergence form equation

$$
\frac{\partial}{\partial y^i}\left(\bar{G}^{ij}(y)\frac{\partial w}{\partial y^j}\right) = C, \qquad (2.16)
$$

where $\bar{G}^{ij}(y) = G^{ij}(y, D\hat{u}(y))$. We now have the inequality that we were hoping for. It is important to note that our choice of κ is determined only by the dimension m, the codimension n, the domain Ω (through dependence on F) and the upper bound Φ_2 on the C^2 norm of ϕ (since the term $|D^3\tilde{\phi}|$ only appears on its own in our inequalities, never multiplied by $|D^2\hat{u}|$).

Lemma 2.4.2. *Let Ω be a smooth, bounded domain in $\mathbb{R}^m$, and let $\phi : \bar{\Omega} \to \mathbb{R}^n$ be a smooth function with $||\phi||_2 \leq \Phi_2$ and $||\phi||_3 \leq \Phi_3$ for some constants $\Phi_2, \Phi_3 > 0$. There exist constants $\kappa, \alpha \in (0, 1)$ and $K > 0$ such that if a maximal graph in $\mathbb{R}^{m+n}_n$ is given by a smooth function $u : \bar{\Omega} \to \mathbb{R}^n$, with $u|_{\partial\Omega} = \phi|_{\partial\Omega}$ and $|||Du|||^2 \leq 1 - \kappa$, then*

$$[Du]_\alpha \leq K.$$

Here κ depends on m, n, Φ_2 and Ω, while α and K depend on m, n, Ω and Φ_3.

This is proved by combining the interior estimates from Lemma 2.4.1 with boundary estimates proved by applying a boundary Harnack inequality to supersolutions of (2.16). This is done exactly as in section 13.4 of [10], and we have seen the most important steps in the proof of Lemma 2.4.1. We will therefore only give a quick outline of the proof.

Proof. Since we choose F such that $F(B \cap \partial\Omega)$ lies in the plane $y^m = 0$, and since we defined $\hat{u}$ such that it is zero on this flat boundary portion, we know that the tangential derivatives $\partial\hat{u}/\partial y^\ell$ will be zero there, and therefore so will w. Using this fact, we apply the boundary weak Harnack inequality (Theorem 8.26 of [10])[27] in balls with centre on $F(B \cap \partial\Omega)$. Using the same ideas as in the proof of Lemma 2.4.1, we get $C^{0,\alpha}$ estimates on the tangential derivatives. Now we just need a $C^{0,\alpha}$ estimate on the normal derivative $\partial\hat{u}/\partial y^m$, which we get from the estimates

[27] Let L be a linear elliptic operator, $Lw = (\partial/\partial x^i)(a^{ij}(x)\partial w/\partial x^j)$ on Ω, with eigenvalues of a^{ij} between two positive constants $\lambda \leq \Lambda$. Let $f \in L^m(\Omega)$ and let $w \in C^2(\bar{\Omega})$ be a supersolution of $Lw = f$ in Ω with $w \geq 0$ in the intersection of Ω with $B^m_{4R}(y) \subset \mathbb{R}^m$. Then $||w^{(-)}||_{L^1(B^m_{2R}(y))} \leq R^m C(\inf_{B^m_R(y)} w^{(-)} + \lambda^{-1}R||f||_{L^m(\Omega)})$, for some constant C depending on m and Λ/λ, where we define $w^{(-)}(x)$ to be equal to $\min(w(x), \inf_{\partial\Omega\cap B^m_{4R}(y)} w)$ when $x \in \Omega$ and equal to $\inf_{\partial\Omega\cap B^m_{4R}(y)} w$ when $x \notin \Omega$.

on the tangential derivatives by using inequality (2.14) (see section 13.4 of [10] for details). Returning to the original domain and boundary data gives $[Du]_\alpha$ estimates (dependent on F) in balls with centre on $\partial\Omega$. Since the domain is bounded, we only need to consider finitely balls (i.e. finitely many transformations F). Then we can take the maximum (over all of the balls and transformations needed) of the constants involved, thus getting the required estimates in a neighbourhood of $\partial\Omega$, with finite constants depending on Ω, etc. Combining with the interior estimates from Lemma 2.4.1 completes the proof. □

Theorem 2.4.1. *Given a convex, smooth, bounded domain Ω in $\mathbb{R}^m$, there will exist a constant C (depending on Ω, m, n) such that the maximal graph Dirichlet problem in $\mathbb{R}^{m+n}_n$ will have a smooth solution, with $u|_{\partial\Omega} = \phi|_{\partial\Omega}$, for any smooth $\phi : \bar{\Omega} \to \mathbb{R}^n$ with C^2 norm less than C.*

Proof. Let $||\phi||_2 \leq \Phi_2$ and $||\phi||_3 \leq \Phi_3$, for constants $\Phi_2, \Phi_3 > 0$. Let $\kappa = \kappa(m, n, \Omega, \Phi_2)$ be as in Lemma 2.4.2. Assume further that $||\phi||_2$ is small enough that $|||D\phi|||^2 \leq 1 - \kappa$ and that inequality (2.3) holds for this κ. This gives the gradient estimate needed to apply Lemma 2.4.2. Then we have an a priori estimate on the $C^{1,\alpha}$ norm. These estimates clearly also hold for solutions with boundary values $\sigma\phi$, for any $\sigma \in [0, 1]$, allowing us to apply Lemma 2.1.1. □

2.5 A Gradient Estimate for Mean Curvature Flow

Soon we will see a situation where we would like to have a gradient bound on graphic solutions to the spacelike mean curvature flow system in $\mathbb{R}^{m+n}_n$. Although this system will be discussed in more detail later, it is convenient to prove a gradient estimate now since the idea is roughly the same as in the proof of gradient estimates for maximal graphs seen in this chapter (also, see [23] and [20] for a similar gradient estimate in the Euclidean case). Another reason for proving this estimate now is that it will give us some confidence that our assumptions in the next chapter are reasonable. More precisely, it provides us with examples where an a priori gradient bound stronger than the spacelike condition will hold (compare to Assumption 3 in the next chapter).

Our goal in this section will be to prove a gradient estimate for spacelike mean curvature flows satisfying certain boundary/initial conditions. Suppose that we have a graphic mean curvature flow in $\mathbb{R}^{m+n}_n$, given by some function $u : \Omega \times (0, T) \to \mathbb{R}^n$ for a bounded, convex, C^2 domain $\Omega \subset \mathbb{R}^m$. We assume u is smooth on the interior of its domain and C^1 on the closure. We take the induced metric from $\mathbb{R}^{m+n}_n$ on spatial slices $M_t = \{(x, u(x,t)) \in \mathbb{R}^{m+n}_n \mid x \in \Omega\}$ for each $t \in [0, T]$, and we assume that these are spacelike (i.e. that $|||Du||| < 1$, where D is taken with respect to the space variables in $\mathbb{R}^m$ only). By the mean curvature flow condition, u satisfies the parabolic system $\partial u/\partial t = g^{ij}(Du)\partial^2 u/\partial x^i \partial x^j$ (we will see why in Theorem 3.3.1).

Proposition 2.5.1. *Let $\phi : \bar{\Omega} \times [0, T] \to \mathbb{R}^n$ be a C^2 function and let $\kappa \in (0, 1)$. If the function u above satisfies the boundary/initial condition that $u(x, t) = \phi(x, t)$ whenever $x \in \partial\Omega$ or $t = 0$, then the inequality $\sup_\Omega |||Du|||^2 < 1 - \kappa$ will hold for all times in $[0, T]$ if the (parabolic) C^2 norm of ϕ is small enough.*

Proof. For ϕ small enough in C^2, we can assume that at time $t = 0$ we have $\sup_\Omega |||Du(\cdot, 0)|||^2 = \sup_\Omega |||D\phi(\cdot, 0)|||^2 < 1 - \kappa^{1/m} < 1 - \kappa$. Suppose that there exists some first time $\epsilon \in (0, T]$ such that $|||Du|||^2 = 1 - \kappa$ for some point in $\bar{\Omega}$. Then we have $|||Du|||^2 \leq 1 - \kappa$ on $\bar{\Omega} \times [0, \epsilon]$. Now we take the linear parabolic operator $L = \partial/\partial t - g^{ij}(Du)\partial^2/\partial x^i \partial x^j$, and we define S and S' exactly as we did earlier (where d is still a function of the space variables on $\bar{\Omega}$ only, independent of the time variable). We get

$$LS = \frac{\nu\zeta^2}{(1+\zeta d)^2} g^{ij} \frac{\partial d}{\partial x^i} \frac{\partial d}{\partial x^j} - L\phi^\gamma \quad \text{and} \quad LS' = \frac{\nu\zeta^2}{(1+\zeta d)^2} g^{ij} \frac{\partial d}{\partial x^i} \frac{\partial d}{\partial x^j} + L\phi^\gamma,$$

and we have

$$\frac{\nu\zeta^2 g^{ij}}{(1+\zeta d)^2} \frac{\partial d}{\partial x^i} \frac{\partial d}{\partial x^j} \geq \frac{\nu\zeta^2}{(1+\zeta \operatorname{diam}\Omega)^2} \quad \text{and} \quad |L\phi^\gamma| \leq \left|\frac{\partial \phi}{\partial t}\right| + \frac{m}{\kappa}|||D^2\phi|||.$$

If we have

$$\frac{\nu\zeta^2}{(1+\zeta \operatorname{diam}\Omega)^2} \geq \left|\frac{\partial \phi}{\partial t}\right| + \frac{m}{\kappa}|||D^2\phi||| \tag{2.17}$$

then we can apply the parabolic maximum principle (see Theorem B.3.1) to the inequalities $LS \geq 0$ and $LS' \geq 0$ to get $S \geq 0$ and $S' \geq 0$, and then a bound on the normal derivative of u at $\partial\Omega$ exactly as in the maximal graph case. This again gives us a bound of the form $|||Du||| \leq \nu\zeta + 2|||D\phi|||$ at $\partial\Omega$, where we can minimize $\nu\zeta$ in such a way that inequality (2.17) holds by taking $\zeta = 1/\text{diam}\Omega$ and $\nu\zeta = 4\text{diam}\Omega \sup_{\Omega\times[0,T]}(|\partial\phi/\partial t| + m|||D^2\phi|||/\kappa)$. Then, for ϕ with small enough parabolic C^2 norm, we will have $|||Du|||^2 < C$ on the parabolic boundary $\partial\Omega \times [0,\epsilon] \cup \Omega \times \{0\}$ for some constant $C < 1 - \kappa^{1/m}$. Since we already know (by the definition of ϵ) that $|||Du|||^2 \leq 1 - \kappa$ on $\bar{\Omega} \times [0,\epsilon]$, we can use

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) \log \sqrt{\det g} \geq 0,$$

which follows from the proof of Proposition 5.2 in [17] (see inequality (A.3)) and where Δ_{M_t} is the induced Laplace operator on M_t.[28] We use this to extend our boundary gradient estimate to all of $\bar{\Omega} \times [0,\epsilon]$. By applying the parabolic maximum principle to this inequality, we see that $|||Du|||^2 < 1 - \kappa$ on $\bar{\Omega} \times [0,\epsilon]$. This is a contradiction to the definition of ϵ, so no such ϵ can exist. Therefore, if the C^2 norm of ϕ is as small as described, the gradient estimate $|||Du|||^2 < 1 - \kappa$ will hold for all times for which this mean curvature flow exists. □

Paying closer attention to this proof gives a more general condition on ϕ which guarantees the existence of such a gradient estimate ($\sup_\Omega |||D\phi(\cdot,0)||| < \sqrt{1-\kappa^{1/m}}$ and $4\text{diam}\Omega \sup_{\Omega\times[0,T]}(|\partial\phi/\partial t| + m|||D^2\phi|||/\kappa) + 2\sup_{\partial\Omega\times[0,T]} |||D\phi||| < \sqrt{1-\kappa^{1/m}}$). It is also worth noting that, along with suitable regularity theorems (see the next two chapters), this estimate could possibly help us to prove long time existence for certain mean curvature flow problems. This would possibly lead to another existence theorem for the maximal graph system Dirichlet problem.

[28] We think of $d/dt - \Delta_{M_t}$ a parabolic operator on $\Omega \times (0,\epsilon)$ since $\Delta_{M_t} f = g^{ij}\partial^2 f/\partial x^i \partial x^j + (1/\sqrt{\det g})(\partial(\sqrt{\det g} g^{ij})/\partial x^j)\partial f/\partial x^i$ and $df/dt = \partial f/\partial t + Df \cdot \partial u/\partial t$, where g^{ij} is positive definite.

Chapter 3

Regularity for Spacelike Mean Curvature Flows

Let $\mathcal{M}$ be a mean curvature flow in a Euclidean space. Let $\mathcal{M}(t)$ be the m-dimensional submanifold of $\mathbb{R}^{m+n}$ given by the flow at each time t. For spacetime points (y, s), the *Gaussian density ratio* of the flow is defined by taking the integral of the backward heat kernel,

$$\Phi(x) = \frac{1}{(4\pi(s-t))^{m/2}} \exp\left(-\frac{|x-y|^2}{4(s-t)}\right),$$

over each $\mathcal{M}(t)$ at times $t < s$. In [12], Huisken proved an important monotonicity formula which roughly says that the Gaussian density ratio will be non-increasing with respect to t on mean curvature flows. A local version of this formula is proved by Ecker in Proposition 4.17 of [5]. One application of these monotonicity formulas is the proof of Brian White's local regularity theorem (see [24]) for mean curvature flows in Euclidean spaces. This theorem says that such a flow will be smooth in regions of spacetime where the Gaussian density ratios are close enough to 1.

Our goal in this chapter is to prove a similar regularity theorem, but now for spacelike mean curvature flows in semi-Euclidean spaces. We will assume that these flows are graphs and that they satisfy some uniform gradient bound stronger than the spacelike condition. Roughly, we will prove that if such a flow is smooth on an interval $(0, T)$, then it can be extended smoothly to time T. This should be compared to Theorem 3.5 of [24]. We prove this by defining a quantity that has similar properties to the Gaussian density ratio. This quantity is chosen in such a way that the evolution equations for spacelike mean curvature flows will allow us to prove monotonicity formulas similar to Huisken's and Ecker's. The proof of the regularity theorem itself is then similar to the proofs in [24] and [5], with some adjustments.

The main differences between this case and the Euclidean case are caused by the semi-Euclidean metric. Obviously, the mean curvature flow system is only parabolic when the spacelike condition is satisfied. Therefore any gradient estimates are only useful if they are stronger than the spacelike condition (for example, the gradient estimate that we proved earlier). This is why we will use Assumption 3. This seems like a significant restriction, but it is not surprising that we need it since most parabolic problems require a gradient estimate anyway. Assumption 3 is also useful when defining our modified version of the Gaussian density ratio. For example, we need a gradient bound to guarantee that this quantity is finite on a smooth flow (since we need the eigenvalues of the induced metric to stay uniformly away from zero). We will frequently need Assumption 3, used with inequality (3.12), to get the uniform bounds needed to use the dominated convergence theorem (such arguments here are more difficult than in the Euclidean case).

Other difficulties due to the semi-Euclidean metric appear in the proofs of the monotonicity and regularity theorems. For example, Ecker's local formula involves a nice localisation function which is not useful in the semi-Euclidean case, thus making our proof of local monotonicity slightly more awkward (see Theorem 3.5.1 and compare to Proposition 4.17 in [5]). We also get different signs in the evolution equations for various quantities, meaning that the inequalities seen in the Euclidean case are often reversed here. Finally, when proving regularity theorems, the metric prevents us from using White's local $C^{2,\alpha}$ norm, since the definition of this norm involves rotations in space. We need to use the spacelike condition, as well as a slightly different version of the Schauder estimates, to avoid the need for rotations.

3.1 Preliminaries

When $N \geq 2$, $\mathbb{R}^N$ will be Euclidean space with elements denoted by x and the usual norm $|x|$. $B^N_R(x)$ will be the ball of radius R and centre x. We will denote by $\mathbb{R}^{N,1}$ the spacetime $\mathbb{R}^N \times \mathbb{R}$ with elements $X = (x,t)$ and parabolic norm $||X|| = \max\{|x|, |t|^{1/2}\}$. We write

$$B^{N,1}_R(X) = B^N_R(x) \times (t - R^2, t + R^2) \quad \text{and} \quad U^{N,1}_R(X) = B^N_R(x) \times (t - R^2, t].$$

The function $\tau : \mathbb{R}^{N,1} \to \mathbb{R}$ will be the projection $\tau(x,t) = t$ onto the time axis. For any $\lambda > 0$, we define the parabolic dilation $D_\lambda : \mathbb{R}^{N,1} \to \mathbb{R}^{N,1}$ by

$$D_\lambda(x,t) = (\lambda x, \lambda^2 t).$$

It is important to notice that $||D_\lambda X|| = \lambda ||X||$. For subsets U of $\mathbb{R}^{N,1}$ and functions f from U into some Euclidean space, we define (as in [24]) the distance

$$d(X,U) = \inf\{||X - Y|| \mid Y \notin U\},$$

and the parabolic Hölder norms (for non-negative integers p and $0 < \alpha < 1$)

$$||f||_{p,\alpha} = ||f||_{C^{p,\alpha}(U)} = \sum_{k+2h \leq p} ||D^k (\partial_t)^h f||_{0,\alpha},$$

where $\partial_t f = \partial f / \partial t$, $\partial_A f = \partial f / \partial x^A$, $D = (\partial_1, \ldots, \partial_N)$,

$$[f]_\alpha = \sup_{X \neq Y \text{ in } U} \frac{|f(X) - f(Y)|}{||X - Y||^\alpha} \quad \text{and} \quad ||f||_{0,\alpha} = \sup_{X \in U} |f(X)| + [f]_\alpha.$$

In the obvious way, we also define the parabolic C^p norm by

$$||f||_p = ||f||_{C^p(U)} = \sum_{k+2h \leq p} \sup_U |D^k (\partial_t)^h f|.$$

If we say that a sequence of functions converges in C^p or $C^{p,\alpha}$ on some set, then we just mean that it converges on that set with respect to the corresponding norm.

For integers $m \geq 2$ and $n \geq 1$, it will be convenient here for us to consider the space $\mathbb{R}^{m+n}$ with elements denoted by $x = (\hat{x}, \tilde{x})$, where $\hat{x} \in \mathbb{R}^m$ and $\tilde{x} \in \mathbb{R}^n$. With this notation, we can write $\mathbb{R}^{m+n}_n = (\mathbb{R}^{m+n}, \langle \cdot, \cdot \rangle)$ with $\langle x, y \rangle = \hat{x} \cdot \hat{y} - \tilde{x} \cdot \tilde{y}$. If we use the summation convention with indices $i, j = 1, \ldots, m$ and $\nu, \gamma = m+1, \ldots, m+n$, then $\langle x, y \rangle = x^i y^i - x^\gamma y^\gamma$ and we denote by $\bar{g}$ the corresponding diagonal matrix with $\bar{g}_{ij} = \delta_{ij}$, $\bar{g}_{\nu\gamma} = -\delta_{\nu\gamma}$.

3.2 Mean Curvature Flows

We will consider subsets $\mathcal{M}$ of the spacetime $\mathbb{R}^{m+n}_n \times \mathbb{R}$, where $\tau(\mathcal{M}) = I$ for some interval I and τ has no critical points. We define the spatial slices $\mathcal{M}(s) = \{x \in \mathbb{R}^{m+n}_n \mid (x,s) \in \mathcal{M}\}$, and assume that each $\mathcal{M}(s)$ is an m-dimensional spacelike submanifold of $\mathbb{R}^{m+n}_n$. We will assume, for each $s \in I$, that there exists some open set U in $\mathbb{R}^{m+n}_n \times I$ such that $\mathcal{M}(s) \subset U$ and

$$\mathcal{M} \cap U = \{(F(\hat{x},t),t) \mid (\hat{x},t) \in \mathcal{E}\},$$

for some open set $\mathcal{E}$ in $\mathbb{R}^m \times I$ and some smooth $F : \mathcal{E} \to \mathbb{R}^{m+n}_n$, where each $F(\cdot,t)$ is an embedding. We call $\mathcal{M}$ a *flow*. If we denote by $H(x,t)$ the mean curvature vector of $\mathcal{M}(t)$ in $\mathbb{R}^{m+n}_n$ at each point x, then we call $\mathcal{M}$ a *mean curvature flow* if each of the functions F above can be chosen to satisfy $\partial_t F(\hat{x},t) = H(F(\hat{x},t),t)$.

Assumption 1: $\mathcal{M}$ is a mean curvature flow of the form above, where each spatial slice is an m-dimensional spacelike submanifold of $\mathbb{R}^{m+n}_n$.

It is not difficult to prove the following facts (note that we will repeatedly use the fact that $\Delta_{\mathcal{M}(t)} F = H$, as proved in Proposition A.1.1). The first is a version of the divergence theorem on mean curvature flows,[1]

$$\begin{aligned}
\int_{\mathcal{M}(t)} \langle H, V\rangle &= \int_{\mathcal{M}(t)} \left\langle \Delta_{\mathcal{M}(t)} F, V\right\rangle \\
&= \int_{\Omega_t} \left\langle \frac{1}{\sqrt{\det g}} \partial_j (\sqrt{\det g} g^{ij} \partial_j F), V \right\rangle \sqrt{\det g} d\hat{x} \\
&= \int_{\Omega_t} \partial_i \left\langle \sqrt{\det g} g^{ij} \partial_j F, V\right\rangle d\hat{x} \\
&\quad - \int_{\Omega_t} \langle \partial_j F, \partial_i V\rangle\, g^{ij} \sqrt{\det g} d\hat{x} \\
&= -\int_{\mathcal{M}(t)} \mathrm{div}_{\mathcal{M}(t)} V
\end{aligned}$$

[1]We use the usual equations for the induced Laplace operator and divergence (see equations (A.1)), and the usual divergence theorem on a domain in $\mathbb{R}^m$: $\int_{\partial\Omega} V \cdot n = \int_\Omega \mathrm{div}_{\mathbb{R}^m} V$ where n is the outward unit normal to $\partial\Omega$.

for vector fields[2] V with compact support on $\mathcal{M}(t)$, where the integrals are taken over $\mathcal{M}(t)$ with respect to the induced metric $g = (g_{ij}) = (\langle \partial_i F, \partial_j F \rangle)$, and where Ω_t is the domain of $F(\cdot, t)$.[3] If $f(x,t)$ is a real valued function defined on $\mathcal{M}$, then

$$\begin{aligned}
\frac{df}{dt} &= \partial_t f + Df \cdot \partial_t F \\
&= \partial_t f + \langle \bar{g} Df, \partial_t F \rangle \\
&= \partial_t f + \langle \bar{g} Df, H \rangle, \qquad (3.1) \\
\Delta_{\mathcal{M}(t)} f &= \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j f) \\
&= \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} Df \cdot \partial_j F) \\
&= \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j F) \cdot Df + g^{ij} \partial_i (Df) \cdot \partial_j F \\
&= \left\langle \Delta_{\mathcal{M}(t)} F, \bar{g} Df \right\rangle + g^{ij} \langle \partial_i (\bar{g} Df), \partial_j F \rangle \\
&= \left\langle \Delta_{\mathcal{M}(t)} F, \bar{g} Df \right\rangle + \operatorname{div}_{\mathcal{M}(t)} (\bar{g} Df) \\
&= \langle H, \bar{g} Df \rangle + \operatorname{div}_{\mathcal{M}(t)} (\bar{g} Df), \qquad (3.2)
\end{aligned}$$

where $\bar{g}$ is the matrix defined in the previous section. The second equation here, along with the divergence theorem above, gives[4]

$$\begin{aligned}
\int_{\mathcal{M}(t)} \left(\phi \Delta_{\mathcal{M}(t)} \eta - \eta \Delta_{\mathcal{M}(t)} \phi \right) &= \int_{\mathcal{M}(t)} \phi \operatorname{div}_{\mathcal{M}(t)} \operatorname{grad}_{\mathcal{M}(t)} \eta - \int_{\mathcal{M}(t)} \eta \operatorname{div}_{\mathcal{M}(t)} \operatorname{grad}_{\mathcal{M}(t)} \phi \\
&= \int_{\mathcal{M}(t)} \left(\operatorname{div}_{\mathcal{M}(t)} (\phi \operatorname{grad}_{\mathcal{M}(t)} \eta) - \left\langle \operatorname{grad}_{\mathcal{M}(t)} \eta, \operatorname{grad}_{\mathcal{M}(t)} \phi \right\rangle \right) \\
&\quad - \int_{\mathcal{M}(t)} \left(\operatorname{div}_{\mathcal{M}(t)} (\eta \operatorname{grad}_{\mathcal{M}(t)} \phi) - \left\langle \operatorname{grad}_{\mathcal{M}(t)} \eta, \operatorname{grad}_{\mathcal{M}(t)} \phi \right\rangle \right) \\
&= - \int_{\mathcal{M}(t)} \left\langle H, \phi \operatorname{grad}_{\mathcal{M}(t)} \eta \right\rangle \\
&\quad + \int_{\mathcal{M}(t)} \left\langle H, \eta \operatorname{grad}_{\mathcal{M}(t)} \phi \right\rangle \\
&= 0 \qquad (3.3)
\end{aligned}$$

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[2]These are not necessarily tangent, since we can still define the divergence by $g^{ij} \langle \partial_j F, \partial_i V \rangle$.

[3]Whenever it will not cause confusion, we will write integrals of the form $\int_{x \in \mathcal{M}(t)} f(x,t) dx$ as $\int_{\mathcal{M}(t)} f$ to save space. Such integrals are *always* taken with respect to the induced metric from $\mathbb{R}^{m+n}_n$. Similarly, we write $\Delta_{\mathcal{M}(t)} f(x,t)$ as $\Delta_{\mathcal{M}(t)} f$ when the meaning is clear.

[4]We use the facts that $\Delta_{\mathcal{M}(t)} = \operatorname{div}_{\mathcal{M}(t)} \operatorname{grad}_{\mathcal{M}(t)}$ and $\operatorname{div}_{\mathcal{M}(t)}(\phi V) = g^{ij} \langle \partial_i F, \nabla_{\partial_j} (\phi V) \rangle = \left\langle g^{ij} \partial_j \phi \partial_i F, V \right\rangle + \phi g^{ij} \langle \partial_i F, \nabla_{\partial_j} V \rangle = \left\langle \operatorname{grad}_{\mathcal{M}(t)} \phi, V \right\rangle + \phi \operatorname{div}_{\mathcal{M}(t)} V$, as well as the fact that the gradient is a tangent vector field, while H is a normal vector field.

whenever ϕ and η are C^2 on $\mathcal{M}(t)$ with ϕ having compact support. Finally, using the usual formula for differentiating determinants, we have the following evolution equation on mean curvature flows,

$$\begin{aligned}
\frac{d}{dt}\sqrt{\det g} &= \frac{1}{2\sqrt{\det g}} g^{ij} \det g \frac{d}{dt} g_{ij} \\
&= \frac{\sqrt{\det g}}{2} g^{ij} \partial_t \langle \partial_i F, \partial_j F \rangle \\
&= \sqrt{\det g} g^{ij} \langle \partial_t \partial_i F, \partial_j F \rangle \\
&= \sqrt{\det g} g^{ij} \langle \partial_i H, \partial_j F \rangle \\
&= \sqrt{\det g} g^{ij} \partial_i \langle H, \partial_j F \rangle - \sqrt{\det g} g^{ij} \langle H, \partial_{ij} F \rangle \\
&= 0 - \sqrt{\det g} \left\langle H, (g^{ij}\partial_{ij} F)^{\perp} \right\rangle \\
&= -\sqrt{\det g} \langle H, H \rangle .
\end{aligned}$$

Definition 3.2.1. *Let $X_0 = (x_0, t_0) \in \mathbb{R}^{m+n,1}$, then we define a function Φ_{X_0} : $\mathbb{R}^{m+n} \times (-\infty, t_0) \to \mathbb{R}$ by*

$$\Phi_{X_0}(x,t) = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(- \frac{\langle x - x_0, x - x_0 \rangle}{4(t_0 - t)} \right).$$

For a flow $\mathcal{M}$ we define

$$\Theta(\mathcal{M}, X_0, t) = \int_{x \in \mathcal{M}(t)} \Phi_{X_0}(x,t),$$

when $t < t_0$.

We see that

$$\begin{aligned}
\frac{\partial \Phi_{X_0}}{\partial t}(x,t) &= \frac{m\Phi_{X_0}(x,t)}{2(t_0 - t)} - \frac{\langle x - x_0, x - x_0 \rangle \Phi_{X_0}(x,t)}{4(t_0 - t)^2}, \qquad (3.4) \\
\bar{g} D \Phi_{X_0}(x,t) &= -\frac{\Phi_{X_0}}{4(t_0 - t)} \bar{g} \cdot D \langle x - x_0, x - x_0 \rangle \\
&= -\frac{\Phi_{X_0}}{4(t_0 - t)} \bar{g} \cdot 2\bar{g}(x - x_0) \\
&= -\frac{(x - x_0)\Phi_{X_0}(x,t)}{2(t_0 - t)}. \qquad (3.5)
\end{aligned}$$

These equations, combined with equations (3.1) and (3.2), give

$$\begin{aligned}\left(\frac{d}{dt}+\Delta_{\mathcal{M}(t)}\right)\Phi_{X_0} &= \partial_t\Phi_{X_0}+2\left\langle \bar{g}D\Phi_{X_0},H\right\rangle+\operatorname{div}_{\mathcal{M}(t)}(\bar{g}D\Phi_{X_0})\\ &= \partial_t\Phi_{X_0}+\operatorname{div}_{\mathcal{M}(t)}(\bar{g}D\Phi_{X_0})+\frac{\left\langle(\bar{g}D\Phi_{X_0})^\perp,(\bar{g}D\Phi_{X_0})^\perp\right\rangle}{\Phi_{X_0}}\\ &\quad -\left\langle H-\frac{(\bar{g}D\Phi_{X_0})^\perp}{\Phi_{X_0}},H-\frac{(\bar{g}D\Phi_{X_0})^\perp}{\Phi_{X_0}}\right\rangle\Phi_{X_0}+\langle H,H\rangle\,\Phi_{X_0}.\end{aligned} \tag{3.6}$$

But the first three terms on the right hand side of this equation add up to 0 since (using equations (3.4) and (3.5))

$$\begin{aligned}\operatorname{div}_{\mathcal{M}(t)}(\bar{g}D\Phi_{X_0}) &= g^{ij}\left\langle\partial_i(\bar{g}D\Phi_{X_0}),\partial_jF\right\rangle\\ &= g^{ij}\left\langle D(\bar{g}D\Phi_{X_0})\cdot\partial_iF,\partial_jF\right\rangle\\ &= \frac{-g^{ij}}{2(t_0-t)}\left\langle D[(x-x_0)\Phi_{X_0}]\cdot\partial_iF,\partial_jF\right\rangle\\ &= \frac{-\Phi_{X_0}g^{ij}}{2(t_0-t)}\left\langle\partial_iF,\partial_jF\right\rangle+\frac{g^{ij}\Phi_{X_0}}{4(t_0-t)^2}\left\langle(x-x_0),\partial_iF\right\rangle\left\langle(x-x_0),\partial_jF\right\rangle\\ &= \frac{-m\Phi_{X_0}}{2(t_0-t)}+\frac{\Phi_{X_0}}{4(t_0-t)^2}\left\langle(x-x_0)^\top,(x-x_0)^\top\right\rangle\end{aligned}$$

and

$$\frac{\left\langle(\bar{g}D\Phi_{X_0})^\perp,(\bar{g}D\Phi_{X_0})^\perp\right\rangle}{\Phi_{X_0}}=\frac{\Phi_{X_0}}{4(t_0-t)^2}\left\langle(x-x_0)^\perp,(x-x_0)^\perp\right\rangle.$$

We now use this, and the evolution equation for $\sqrt{\det g}$, to differentiate

$$\int_{x\in\mathcal{M}(t)}\Phi_{X_0}(x,t)\phi(x,t)$$

when ϕ is some non-negative C^2 function with each $\phi(\cdot,t)$ having compact support on $\mathcal{M}(t)$. First we see that

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}(t)}\Phi_{X_0}(x,t)\phi(x,t) &= \frac{d}{dt}\int_{\Omega_t}\Phi_{X_0}(F(\hat{x},t),t)\phi(F(\hat{x},t),t)\sqrt{\det g}\,d\hat{x}\\ &= \int_{\Omega_t}\partial_t(\Phi_{X_0}(F(\hat{x},t),t)\phi(F(\hat{x},t),t)\sqrt{\det g})d\hat{x}\\ &= \int_{\Omega_t}\left(\phi\frac{d}{dt}\Phi_{X_0}+\Phi_{X_0}\frac{d}{dt}\phi-\Phi_{X_0}\phi\left\langle H,H\right\rangle\right)\sqrt{\det g}\,d\hat{x}\\ &= \int_{\mathcal{M}(t)}\left(\frac{d\Phi_{X_0}}{dt}\phi+\Phi_{X_0}\frac{d\phi}{dt}-\Phi_{X_0}\phi\left\langle H,H\right\rangle\right),\end{aligned}$$

and then

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}(t)}\Phi_{X_0}\phi &= \int_{\mathcal{M}(t)}\left(\phi\frac{d\Phi_{X_0}}{dt}+\Phi_{X_0}\frac{d\phi}{dt}-\langle H,H\rangle\,\phi\Phi_{X_0}\right)\\
&= \int_{\mathcal{M}(t)}\Phi_{X_0}\left(\frac{d}{dt}-\Delta_{\mathcal{M}(t)}\right)\phi+\left(\left(\frac{d}{dt}+\Delta_{\mathcal{M}(t)}\right)\Phi_{X_0}-\langle H,H\rangle\,\Phi_{X_0}\right)\phi\\
&= \int_{\mathcal{M}(t)}\Phi_{X_0}\left(\frac{d}{dt}-\Delta_{\mathcal{M}(t)}\right)\phi-\left\langle H-\frac{(\bar{g}D\Phi_{X_0})^{\perp}}{\Phi_{X_0}},H-\frac{(\bar{g}D\Phi_{X_0})^{\perp}}{\Phi_{X_0}}\right\rangle\phi\Phi_{X_0},\end{aligned}$$

where in the second step we used equation (3.3) and the last step uses equation (3.6). By (3.5) this gives:

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}(t)}\Phi_{X_0}\phi \;&=\; \int_{\mathcal{M}(t)}\Phi_{X_0}\left(\frac{d}{dt}-\Delta_{\mathcal{M}(t)}\right)\phi\\
&\quad-\int_{\mathcal{M}(t)}\left\langle H-\frac{(x-x_0)^{\perp}}{2(t_0-t)},H-\frac{(x-x_0)^{\perp}}{2(t_0-t)}\right\rangle\phi\Phi_{X_0}.\end{aligned}\tag{3.7}$$

This will be very useful later, and it is our first step towards the proof of monotonicity formulas. It is important to remember that the second term on the right hand side is non-negative (since the flow is spacelike, which means that normal vectors will be timelike or zero).

Proposition 3.2.1. *Let $X,Y\in\mathbb{R}^{m+n,1}$, $s<\tau(Y)$ and $\lambda>0$, then*

$$\Theta(D_\lambda(\mathcal{M}-X),Y,s)=\Theta(\mathcal{M},X+D_{1/\lambda}Y,\tau(X)+s/\lambda^2).\tag{3.8}$$

Proof. Let $\mathcal{M}$ be given by F near the time $t+s/\lambda^2$, where $F(\cdot,t+s/\lambda^2)$ has domain Ω. If $X=(x,t)=(\hat{x},\tilde{x},t)$, then the flow $D_\lambda(\mathcal{M}-X)$ is given by the function $F_{\lambda,X}(\cdot,\cdot)=\lambda(F(\cdot/\lambda+\hat{x},\cdot/\lambda^2+t)-x)$ near the time s. Obviously $DF_{\lambda,X}(\cdot,\cdot)=DF(\cdot/\lambda+\hat{x},\cdot/\lambda^2+t)$. Now, for $Y=(y,r)=(\hat{y},\tilde{y},r)$, we see that $\Theta(D_\lambda(\mathcal{M}-X),Y,s)$ is equal to

$$\begin{aligned}&\int_{\lambda(\Omega-\hat{x})}\frac{\exp\left(\frac{-\left\langle F_{\lambda,X}(\hat{z},s)-y,F_{\lambda,X}(\hat{z},s)-y\right\rangle}{4(r-s)}\right)}{(4\pi(r-s))^{m/2}}\sqrt{\det DF^T_{\lambda,X}\bar{g}DF_{\lambda,X}}|_{(\hat{z},s)}d\hat{z}\\
=&\int_{\lambda(\Omega-\hat{x})}\left(\frac{\exp\left(\frac{-\langle\lambda F-\lambda x-y,\lambda F-\lambda x-y\rangle}{4(r-s)}\right)}{(4\pi(r-s))^{m/2}}\sqrt{\det DF^T\bar{g}DF}\right)|_{(\hat{z}/\lambda+\hat{x},s/\lambda^2+t)}d\hat{z}\\
=&\int_{\Omega}\left(\lambda^m\frac{\exp\left(\frac{-\lambda^2\langle F-x-y/\lambda,F-x-y/\lambda\rangle}{4(r-s)}\right)}{(4\pi(r-s))^{m/2}}\sqrt{\det DF^T\bar{g}DF}\right)|_{(\hat{z},s/\lambda^2+t)}d\hat{z}\end{aligned}$$

$$= \int_{z\in\mathcal{M}(s/\lambda^2+t)} \frac{\exp\left(\frac{-\langle z-(x+y/\lambda),z-(x+y/\lambda)\rangle}{4(r-s)/\lambda^2}\right)}{(4\pi(r-s)/\lambda^2)^{m/2}}$$
$$= \Theta(\mathcal{M}, X + D_{1/\lambda}Y, t + s/\lambda^2),$$

where we have applied the transformation formula for integrals,[5] using a transformation $\zeta : \Omega \to \lambda(\Omega - \hat{x})$ with $\zeta(\hat{z}) = \lambda(\hat{z} - \hat{x})$. □

Proposition 3.2.2. *Let $\mathcal{M}$ be a spacelike mean curvature flow, as in Assumption 1, with $I = (-\infty, 0]$ and such that each spatial slice $\mathcal{M}(t)$ is a graph over $\Omega = \mathbb{R}^m$. If we have*

$$H(x,t) = \frac{x^\perp}{2t} \tag{3.9}$$

for every point (x,t) on the flow, then $\mathcal{M}$ is invariant under parabolic dilations.

Proof. The idea (as for a similar result in [13]) is to assume that there is some point $Y = (y,t)$ on $\mathcal{M}$ but not on $D_\lambda\mathcal{M}$ for some λ. We then take a compactly supported C^2 function ϕ with $\phi(y) = 1$ and $\phi = 0$ on $D_\lambda\mathcal{M}(t)$. Let $\mathcal{M}$ be given by a function $F(\cdot,\cdot)$ near t, as usual, then

$$\begin{aligned}\int_{D_\lambda\mathcal{M}(t)} \phi &= \int_{\mathbb{R}^m} \phi(\lambda F(\hat{x}/\lambda, t/\lambda^2))\sqrt{\det g}|_{(\hat{x}/\lambda,t/\lambda^2)}d\hat{x}\\ &= \lambda^m \int_{\mathbb{R}^m} \phi(\lambda F(\hat{x}, t/\lambda^2))\sqrt{\det g}|_{(\hat{x},t/\lambda^2)}d\hat{x}\\ &= \lambda^m \int_{\mathcal{M}(t/\lambda^2)} \phi(\lambda x),\end{aligned}$$

where we have again used the transformation formula for integrals, taking a transformation $\hat{x} \mapsto \lambda\hat{x}$. But our evolution equation for $\sqrt{\det g}$ gives

$$\partial_\lambda(\sqrt{\det g}|_{(\hat{x},t/\lambda^2)}) = \left(\frac{2t}{\lambda^3}\langle H, H\rangle \sqrt{\det g}\right)|_{(\hat{x},t/\lambda^2)}$$

and we easily see that

$$\partial_\lambda(\phi(\lambda F(\hat{x},t/\lambda^2))\lambda^m) = m\lambda^{m-1}\phi(\lambda F(\hat{x},t/\lambda^2)) + \lambda^m D\phi(\lambda F)\cdot\left(F - \frac{2t}{\lambda^2}\partial_t F\right)|_{(\hat{x},t/\lambda^2)}.$$

[5] If $\zeta : U \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^m$ and $f : V \to \mathbb{R}$ is integrable, then $\int_{\zeta(U)} f(y)dy = \int_U f(\zeta(x))|\det D\zeta(x)|dx$. See Theorem 18.2 in [9].

These give us

$$\begin{aligned}\frac{d}{d\lambda}\int_{D_\lambda\mathcal{M}(t)}\phi &= \lambda^m\int_{\mathcal{M}(t/\lambda^2)}[\frac{2t}{\lambda^3}\phi(\lambda x)\langle H,H\rangle+\frac{m}{\lambda}\phi(\lambda x)\\ &\qquad +D\phi(\lambda x)\cdot x-\frac{2t}{\lambda^2}D\phi(\lambda x)\cdot H]\\ &= \lambda^m\int_{\mathcal{M}(t/\lambda^2)}[\frac{2t}{\lambda^3}\phi(\lambda x)\left\langle H,\frac{x^\perp}{2t/\lambda^2}\right\rangle\\ &\qquad +D\phi(\lambda x)\cdot x-\frac{2t}{\lambda^2}D\phi(\lambda x)\cdot H+\frac{m}{\lambda}\phi(\lambda x)],\end{aligned}$$

where we have used equation (3.9) to get $H = x^\perp/(2t/\lambda^2)$ on $\mathcal{M}(t/\lambda^2)$, and the fact that $\partial_t F = H$. Now we can deal with the first term of the right hand side of this equation by using the divergence theorem,

$$\int_{\mathcal{M}(t/\lambda^2)}\left\langle H,\phi(\lambda x)x^\perp\right\rangle=\int_{\mathcal{M}(t/\lambda^2)}\langle H,\phi(\lambda x)x\rangle=-\int_{\mathcal{M}(t/\lambda^2)}\operatorname{div}_{\mathcal{M}(t/\lambda^2)}(\phi(\lambda x)x),$$

where we have used the fact that H is a normal vector. So

$$\begin{aligned}\frac{d}{d\lambda}\int_{\mathcal{M}(t/\lambda^2)}\phi &= \lambda^m\int_{\mathcal{M}(t/\lambda^2)}[-\frac{1}{\lambda}\operatorname{div}_{\mathcal{M}(t/\lambda^2)}(\phi(\lambda x)x)+D\phi(\lambda x)\cdot x\\ &\qquad -\frac{2t}{\lambda^2}D\phi(\lambda x)\cdot H+\frac{m}{\lambda}\phi(\lambda x)]\\ &= \lambda^m\int_{\mathcal{M}(t/\lambda^2)}[-\frac{1}{\lambda}\operatorname{div}_{\mathcal{M}(t/\lambda^2)}(\phi(\lambda x)x)+D\phi(\lambda x)\cdot x^\top+\frac{m}{\lambda}\phi(\lambda x)],\end{aligned} \tag{3.10}$$

where we have again substituted $H = x^\perp/(2t/\lambda^2)$ in the last step. Finally we note that, in terms of F, the function $\operatorname{div}_{\mathcal{M}(t/\lambda^2)}(\phi(\lambda x)x)$ is given by

$$\begin{aligned}g^{ij}\langle\partial_i(\phi(\lambda F)F),\partial_jF\rangle|_{(\hat{x},t/\lambda^2)} &= g^{ij}\langle\phi(\lambda F)\partial_iF+(\lambda D\phi(\lambda F)\cdot\partial_iF)F,\partial_jF\rangle|_{(\hat{x},t/\lambda^2)}\\ &= (\phi(\lambda F)g^{ij}g_{ij}+\lambda D\phi(\lambda F)\cdot\underbrace{(\partial_iF)g^{ij}\langle F,\partial_jF\rangle}_{F^\top})|_{(\hat{x},t/\lambda^2)}\\ &= \big(m\phi(\lambda F)+\lambda D\phi(\lambda F)\cdot F^\top\big)|_{(\hat{x},t/\lambda^2)}.\end{aligned}$$

We substitute $\operatorname{div}_{\mathcal{M}(t/\lambda^2)}(\phi(\lambda x)x) = m\phi(\lambda x)+\lambda D\phi(\lambda x)\cdot x^\top$ into equation (3.10) to get

$$\frac{d}{d\lambda}\int_{D_\lambda\mathcal{M}(t)}\phi=0.$$

So $\int_{D_\lambda\mathcal{M}}\phi$ remains constant as λ varies, which is a contradiction and proves our claim. $\square$

3.3 Graphs

We continue to consider flows $\mathcal{M}$ satisfying Assumption 1, but now we add:

Assumption 2: $\mathcal{M}$ is a graph over $\Omega \times I$,

$$\mathcal{M} = \{(\hat{x}, u(\hat{x}, t), t) \mid (\hat{x}, t) \in \Omega \times I\},$$

for a domain Ω in $\mathbb{R}^m$, interval I in $\mathbb{R}$, and smooth function $u : \Omega \times I \to \mathbb{R}^n$.

When we say that such a flow $\mathcal{M}$ is smooth (or locally $C^{2,\alpha}$, etc.), we mean that the function u is smooth (or locally $C^{2,\alpha}$, etc.). We will also discuss sequences $\mathcal{M}_J$ of such flows (where $J = 1, 2, \ldots$). When we talk about convergence of $\mathcal{M}_J$ in some space of functions, we actually mean convergence of the corresponding u_J.

Proposition 3.3.1. *If $\mathcal{M}$ satisfies Assumptions 1 and 2, then the function u will be a solution to the quasilinear parabolic system of equations*

$$\partial_t u = \hat{g}^{ij}(Du)\partial_{ij} u$$

on $\Omega \times I$, where $\hat{g}_{ij} = \delta_{ij} - \partial_i u^\nu \partial_j u^\nu$.

Remember that the system here will be parabolic because the spacelike condition implies that $\hat{g} = I - Du^T Du$ will be positive definite.

Proof. With F as in Assumption 1, we write $F = (\hat{F}, \tilde{F})$ such that $F(\hat{x}, t) = (\hat{F}(\hat{x}, t), u(\hat{F}(\hat{x}, t), t))$. Differentiating with respect to t, using the chain rule,

$$\partial_t F(\hat{x}, t) = (I, Du(\hat{F}(\hat{x}, t), t)) \cdot \partial_t \hat{F}(\hat{x}, t) + (0, \partial_t u(\hat{F}(\hat{x}, t), t)),$$

where the first term on the right hand side is a tangent vector. Therefore (since we know that the left hand side is equal to H, which is normal) we have $\partial_t F(\hat{x}, t) = (0, \partial_t u)^\perp|_{(\hat{F}(\hat{x},t),t)}$. We already know that $\partial_t F(\hat{x}, t) = H(F(\hat{x}, t), t)$, but the mean curvature at $F(\hat{x}, t)$ is given by $(0, \hat{g}^{ij}\partial_{ij}u)^\perp|_{(\hat{F}(\hat{x},t),t)}$ (see the proof of Proposition A.1.1). Hence $(0, \partial_t u)^\perp = (0, \hat{g}^{ij}\partial_{ij}u)^\perp$. Since $(0, \partial_t u - \hat{g}^{ij}\partial_{ij}u)^\perp = 0$ implies that $(0, \partial_t u - \hat{g}^{ij}\partial_{ij}u)$ is a tangent vector, we can write $(0, \partial_t u - \hat{g}^{ij}\partial_{ij}u) = v^k \partial_k(\hat{x}, u) = v^k(e_k, \partial_k u)$. Clearly then each $v^k = 0$, which implies that $(0, \partial_t u - \hat{g}^{ij}\partial_{ij}u) = 0$. $\square$

Proposition 3.3.2. *Let Ω be a domain in $\mathbb{R}^m$ and let I be an open interval in $\mathbb{R}$. If $u : \Omega \times I \to \mathbb{R}^n$ is a smooth solution to the system from Proposition 3.3.1, and if the graph $\mathcal{M}$ of u (as in Assumption 2) gives a spacelike flow (with respect to the induced metric from $\mathbb{R}^{m+n}_n$ on spatial slices), then $\mathcal{M}$ is a mean curvature flow (i.e. Assumption 1 is satisfied).*

Proof. For each $s \in I$, we would like to find an open set $\mathcal{E} \subset \Omega \times I$ containing $\Omega \times \{s\}$, and a function $\phi : \mathcal{E} \to \Omega$ such that $F(\hat{x}, t) = (\phi(\hat{x}, t), u(\phi(\hat{x}, t), t))$ satisfies $\partial_t F(\hat{x}, t) = H(F(\hat{x}, t), t)$. But we know that the mean curvature of our graph is $(0, \hat{g}^{ij}\partial_{ij}u)^\perp$ and that $\partial_t u = \hat{g}^{ij}\partial_{ij}u$. These facts and the chain rule applied to F imply that we need $\partial_t F = (\partial_t\phi, Du\partial_t\phi) + (0, \partial_t u)$ to be equal to $(0, \partial_t u)^\perp$. This is equivalent to $(\partial_t\phi, Du\partial_t\phi) = -\langle(0, \partial_t u), (e_i, \partial_i u)\rangle\, \hat{g}^{ij}(e_j, \partial_j u) = \partial_t u \cdot \partial_i u \hat{g}^{ij}(e_j, \partial_j u)$, which means that we want a solution to the system $\partial_t\phi^j = \partial_t u \cdot \partial_j u \hat{g}^{ij}(Du)|_{(\phi(\hat{x},t),t)}$ for $j = 1, \ldots, m$. Denoting the right hand side of this system by G, we can write this as $\partial_t\phi(\hat{x}, t) = G(\phi(\hat{x}, t), t)$, where G is smooth (since u is). We prove the existence of a solution to this system by considering the nonautonomous (time dependent) system of ordinary differential equations given by

$$\frac{d\phi}{dt}(t) = G(\phi(t), t),$$

with initial condition $\phi(s) = \hat{x}$ for any $\hat{x} \in \Omega$. By the usual existence and uniqueness theorems for such systems,[6] solutions $\phi_{\hat{x},s}(t)$ will exist for each $\hat{x} \in \Omega$ and $s \in I$. Writing $\phi_{\hat{x},s}(t) = \phi_s(\hat{x}, t)$, we see that $\phi_s(\cdot, s)$ is the identity map, ϕ_s is defined on some open set $\mathcal{E}$ containing $\Omega \times \{s\}$, and each $\phi_s(\cdot, t)$ will be a diffeomorphism. Therefore ϕ_s is the required function, so Assumption 1 is satisfied. $\square$

It will be convenient for us to again use the norm $|||Du||| = \sup_{|v|=1} |Du \cdot v|$ for the differential map $Du(\hat{x}, t) : \mathbb{R}^m \to \mathbb{R}^n$.

[6] See Theorem 17.15 and Problem 17-15 of [15]. Suppose that Ω is a domain in $\mathbb{R}^m$, J is an open interval in $\mathbb{R}$ and $V : J \times \Omega \to \mathbb{R}^m$ is a smooth vector field. Then there exists an open set $\mathcal{E}$ and a smooth map $\theta : \mathcal{E} \to \Omega$ such that $\gamma(t) = \theta(t, s, p)$ is the unique maximal solution of the initial value problem $d\gamma/dt = V(t, \gamma(t))$ with $\gamma(s) = p$. Let $(t, s) \in J \times J$ and define $\Omega_{t,s} = \{p \in \Omega \mid (t, s, p) \in \mathcal{E}\}$, then $\theta(t, s, \cdot) : \Omega_{t,s} \to \Omega_{s,t}$ is a diffeomorphism with inverse $\theta(s, t, \cdot)$. Also, see [1] for similar results.

Assumption 3: With $\mathcal{M}$ as in Assumption 2, the function u satisfies $|||Du|||^2 \leq 1-\kappa$ for some $\kappa > 0$ and the domain Ω is smooth and convex.

Of course, the assumption on $|||Du|||$ would follow from any suitable a priori gradient estimate for such flows. When this assumption is satisfied, the eigenvalues of the matrix $\hat{g}$ will always be between κ and 1. It is also worth remembering that this gradient bound is preserved under parabolic dilations of the flow.[7]

With this assumption, we get the following inequality for any $t \in I$,[8]

$$|u(\hat{x},t) - u(\hat{y},t)| \leq \sup_{\Omega} |||Du(\cdot,t)|||.|\hat{x}-\hat{y}| \leq (1-\kappa)^{1/2}|\hat{x}-\hat{y}|, \tag{3.11}$$

and then

$$\begin{aligned} |u(\hat{x},t) - u(\hat{y},s)| &\leq |u(\hat{x},t) - u(\hat{y},t)| + |u(\hat{y},t) - u(\hat{y},s)| \\ &\leq (1-\kappa)^{1/2}|\hat{x}-\hat{y}| + (s-t)\sup_{(t,s)} |\partial_t u(\hat{y},\cdot)| \end{aligned} \tag{3.12}$$

whenever $s \geq t$ are both in I.

Proposition 3.3.3. *Suppose that $u : \Omega \times [a,b) \to \mathbb{R}^n$ is smooth with $|||Du|||^2 \leq 1-\kappa$ and satisfies the system from Proposition 3.3.1. Then u can be extended to a continuous function on $\Omega \times [a,b]$.*

Proof. Take the linear operator $P = \partial_t - \hat{g}^{ij}(Du)\partial_{ij}$ (where we have positive upper and lower bounds on the eigenvalues of g^{ij}, by the bound on Du). Using $Pu = 0$, Theorem 2.14 of [18] (in particular, the comment that follows it)[9] on cylinders in $\Omega \times (a,b)$ tells us that, for any $\hat{x} \in \Omega$, the function $u(\hat{x},\cdot)$ is uniformly continuous

[7] This is obviously true since, if u_λ is the function corresponding to the dilation (by $\lambda > 0$) of the graph of u, then $Du_\lambda(\cdot,\cdot) = D(\lambda u(\cdot/\lambda, \cdot/\lambda^2)) = Du(\cdot/\lambda, \cdot/\lambda^2)$.

[8] To prove this, let Ω be open in $\mathbb{R}^m$, let $u : \Omega \to \mathbb{R}^n$ and let $\hat{x} \in \Omega$ and $h \in \mathbb{R}^m$ be such that $\hat{x}+\delta h \in \Omega$ for all $\delta \in [0,1]$. Then $|u(\hat{x}+h)-u(\hat{x})| = |(\int_0^1 Du(\hat{x}+\delta h)d\delta)\cdot h| \leq \int_0^1 |Du(\hat{x}+\delta h)\cdot h| d\delta = |h| \int_0^1 |Du(\hat{x}+\delta h)\cdot h|/|h| d\delta \leq |h| \sup_\Omega |||Du|||$.

[9] This comment says that, if u satisfies a linear parabolic equation where the coefficient matrix has positive upper and lower bounds on its eigenvalues, and if Du is bounded in some cylinder $\{|\hat{x}-\hat{x}_0| < \rho\} \times (t_1, t_1+\rho^2)$, then $u(\hat{x}_0,\cdot)$ will be uniformly $C^{0,1/2}$ with respect to the time variable.

on some interval with supremum b. It can therefore be extended continuously to $[a, b]$. On $\Omega \times [a, b)$ we know that $|u(\hat{x}, t) - u(\hat{y}, t)| \leq (1-\kappa)^{1/2}|\hat{x} - \hat{y}|$. Take the limit of this inequality as $t \to b$ to see that it holds on all $[a, b]$, so the extension is continuous with respect to $\hat{x} \in \Omega$ since $|u(\hat{x}, t) - u(\hat{y}, t)| \leq (1-\kappa)^{1/2}|\hat{x} - \hat{y}| < \epsilon$ whenever $|\hat{x} - \hat{y}| < \delta = \epsilon/(1-\kappa)^{1/2}$. □

3.4 Monotonicity for Entire Flows

In this section our flows will satisfy Assumptions 1, 2 and 3, but will also be *entire flows* (in other words, the spatial slices will be graphs defined over all of $\mathbb{R}^m$).

Assumption 4: With $\mathcal{M}$ as in Assumption 2, $\Omega = \mathbb{R}^m$ and $I = (-\infty, T]$ for some $T \in (-\infty, \infty]$.

If $\mathcal{M}$ is such an entire flow, it is easy to check that $\Theta(\mathcal{M}, X_0, t)$ is finite at points $X_0 = (\hat{x}_0, u(\hat{x}_0, t_0), t_0)$ on $\mathcal{M}$ for times $t < t_0$. We know that $\sqrt{\det \hat{g}} < 1$. We also have inequality (3.12) which gives us a bound on the exponent in Φ_{X_0} on the flow in terms of $|\hat{x} - \hat{x}_0|$ and finite constants,

$$\begin{aligned}
& -\frac{\langle x - x_0, x - x_0\rangle}{4(t_0 - t)} \\
= & -\frac{\langle(\hat{x}, u(\hat{x}, t)) - (\hat{x}_0, u(\hat{x}_0, t_0)), (\hat{x}, u(\hat{x}, t)) - (\hat{x}_0, u(\hat{x}_0, t_0))\rangle}{4(t_0 - t)} \\
= & \frac{-|\hat{x} - \hat{x}_0|^2 + |u(\hat{x}, t) - u(\hat{x}_0, t_0)|^2}{4(t_0 - t)} \\
\leq & \frac{-|\hat{x} - \hat{x}_0|^2 + \left((1-\kappa)^{1/2}|\hat{x} - \hat{x}_0| + (t_0 - t)\sup_{(t,t_0)}|\partial_t u(\hat{x}_0, \cdot)|\right)^2}{4(t_0 - t)} \\
= & \frac{-\kappa|\hat{x} - \hat{x}_0|^2 + 2(1-\kappa)^{1/2}(t_0 - t)\sup_{(t,t_0)}|\partial_t u(\hat{x}_0, \cdot)||\hat{x} - \hat{x}_0|}{4(t_0 - t)} \\
& + \frac{(t_0 - t)^2 \sup_{(t,t_0)}|\partial_t u(\hat{x}_0, \cdot)|^2}{4(t_0 - t)}.
\end{aligned} \tag{3.13}$$

Here we can use the fact that the flow is smooth, so the time derivative in this inequality will be bounded on (t, t_0) by some constant, and the fact that $t < t_0$ being fixed means that $4(t_0 - t)$ will be just a positive constant. This means that, for large $|\hat{x} - \hat{x}_0|$, the first term in the right hand side of equation (3.13) will dominate. So we have a bound on $\Theta(\mathcal{M}, X_0, t)$ by some integral which is clearly finite.[10]

The simplest example of the kind of flow that we consider in this chapter is a *non-moving plane*, where each spatial slice is equal to a spacelike plane (independent of time). Then Du is constant and $\partial_t u = 0$. Obviously this implies that

$$\begin{aligned} |u(\hat{x}, t) - u(\hat{x}_0, t_0)|^2 &= |Du \cdot \hat{x} - Du \cdot \hat{x}_0|^2 \\ &= |Du \cdot (\hat{x} - \hat{x}_0)|^2 \\ &= (\hat{x} - \hat{x}_0)^T Du^T Du(\hat{x} - \hat{x}_0), \end{aligned}$$

where we know that $\hat{g} = I - Du^T Du$. For any point $X_0 = (\hat{x}_0, u(\hat{x}_0, t_0), t_0)$ on the flow, we therefore see that the exponent of Φ_{X_0} on the flow will involve

$$\begin{aligned} -|\hat{x} - \hat{x}_0|^2 + |u(\hat{x}, t) - u(\hat{x}_0, t)|^2 &= -(\hat{x} - \hat{x}_0)^T (I - Du^T Du)(\hat{x} - \hat{x}_0) \\ &= -(\hat{x} - \hat{x}_0)^T \hat{g}(\hat{x} - \hat{x}_0), \end{aligned}$$

which gives

$$\begin{aligned} \Theta(\mathcal{M}, X_0, t) &= \int_{\mathbb{R}^m} \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(-\frac{(\hat{x} - \hat{x}_0)^T \hat{g}(\hat{x} - \hat{x}_0)}{4(t_0 - t)} \right) \sqrt{\det \hat{g}} d\hat{x} \\ &= \frac{\sqrt{\det \hat{g}}}{(4\pi(t_0 - t))^{m/2}} \sqrt{\frac{(2\pi)^m}{\det(\hat{g}/2(t_0 - t))}} \\ &= 1, \end{aligned}$$

where we again use the usual Gaussian integral formula.

Proposition 3.4.1. Θ *is equal to* 1 *on non-moving planes.*

Proof. As above. □

The following theorem gives us a monotonicity formula, similar to Huisken's, for entire spacelike mean curvature flows. Roughly, it tells us that Θ will be non-decreasing with respect to the time variable on such flows.[11]

---

[10] It is clear that this integral is finite from the usual formula for Gaussian integrals, $\int_{\mathbb{R}^m} \exp(-A_{ij} y^i y^j / 2) dy = \sqrt{(2\pi)^m / \det(A_{ij})}$, where the matrix A_{ij} is constant, symmetric and positive definite. Almost all of the bounds on integrals that we use in the future will follow from this formula.

[11] This is different to the Euclidean case, where the Gaussian density ratio would be non-increasing.

Theorem 3.4.1. *Let $\mathcal{M}$ be a mean curvature flow satisfying Assumptions 1, 2, 3, 4, and let the mean curvature H be bounded on $\mathcal{M}$. Then*

$$\frac{d}{dt}\Theta(\mathcal{M},X_0,t) = -\int_{x\in\mathcal{M}(t)} \left\langle H(x,t) + \frac{(x-x_0)^\perp}{2(t_0-t)}, H(x,t) + \frac{(x-x_0)^\perp}{2(t_0-t)} \right\rangle \Phi_{X_0}(x,t),$$

when $X_0 = (x_0,t_0) \in \mathcal{M}$ and $t < t_0$.

Proof. For each $R > 0$ we can choose[12] functions $\chi_R^m : \mathbb{R}^m \to \mathbb{R}$ such that[13]

$$\chi_{B_R^m(0)} \leq \chi_R^m \leq \chi_{B_{2R}^m(0)} \quad \text{and} \quad R|D\chi_R^m| + R^2|D^2\chi_R^m| \leq C$$

for some constant C. Using these functions, we define $\chi_R : \mathbb{R}_n^{m+n} \to \mathbb{R}$ by taking

$$\chi_R(x) = \chi_R(\hat{x},\tilde{x}) = \chi_R^m(\hat{x})$$

for any $x = (\hat{x},\tilde{x})$. We now apply equation (3.7) with $\phi = \chi_R$ to get

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{M}(t)} \Phi_{X_0}\chi_R &= \int_{\mathcal{M}(t)} \Phi_{X_0}\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)\chi_R \\ &\quad - \int_{\mathcal{M}(t)} \left\langle H + \frac{(x-x_0)^\perp}{2(t_0-t)}, H + \frac{(x-x_0)^\perp}{2(t_0-t)} \right\rangle \Phi_{X_0}\chi_R. \end{aligned} \tag{3.14}$$

Using equations (3.1) and (3.2), the Schwarz inequality and the bounds on the eigenvalues of $\hat{g}$ (from the assumed bound on the gradient), we have[14]

$$\begin{aligned} \left|\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)\chi_R\right| &= |\partial_t\chi_R - \mathrm{div}_{\mathcal{M}(t)}(\bar{g}D\chi_R)| \\ &= |0 - \hat{g}^{ij}\langle \partial_i(\bar{g}D\chi_R), \partial_j(\hat{x},u)\rangle| \\ &= |\hat{g}^{ij}\partial_i(\hat{x},u)\cdot D^2\chi_R \cdot \partial_j(\hat{x},u)^T| \\ &= |\hat{g}^{ij}(Du)\partial_{ij}\chi_R^m| \\ &\leq \sqrt{\sum_{ij}\hat{g}^{ij}(Du)^2}\sqrt{\sum_{ij}(\partial_{ij}\chi_R^m)^2} \\ &= |\hat{g}^{-1}(Du)|\cdot|D^2\chi_R^m| \\ &\leq C_0(\kappa)\frac{C}{R^2}\chi_{B_{2R}^m(0)-B_R^m(0)}, \end{aligned} \tag{3.15}$$

[12]See the proof of Theorem 4.13 in [5], for example.

[13]For a set K, we denote by χ_K the characteristic function of K.

[14]Note that, from now on, $C(\cdot,\ldots,\cdot)$ will always denote a positive constant depending on the quantities in parentheses.

where we have also used the facts that $|\hat{g}^{-1}(Du)| \leq C_0(\kappa)$, that $\langle \bar{g}v, w\rangle = v \cdot w$, and that χ_R^m is constant outside $B_{2R}^m(0) - B_R^m(0)$.

Now we will restrict to any fixed bounded time interval $I' = [a, b] \subset (-\infty, t_0)$, considering only times $t \in I'$. The first thing to note here is that

$$a \leq t \leq b \quad \Rightarrow t_0 - b \leq t_0 - t \leq t_0 - a \quad \Rightarrow \frac{1}{t_0 - a} \leq \frac{1}{t_0 - t} \leq \frac{1}{t_0 - b},$$

so we have uniform upper and lower bounds, independent of t, on $t_0 - t$ and $1/(t_0 - t)$.

Next we note that the flow is smooth on $(-\infty, t_0]$ (by our assumptions in the statement of the theorem) and $X_0 = (\hat{x}_0, u(\hat{x}_0, t_0), t_0)$ lies on the flow, so we have

$$\sup_{[t,t_0]} |\partial_t u(\hat{x}_0, \cdot)| \leq \sup_{[a,t_0]} |\partial_t u(\hat{x}_0, \cdot)|,$$

where $\sup_{[a,t_0]} |\partial_t u(\hat{x}_0, \cdot)|$ is a finite constant independent of $t \in I'$ (but dependent on I' and $\hat{x}_0$, which are fixed). We can use this to apply inequality (3.12) to bound the exponent of Φ_{X_0} on our flow, getting

$$\begin{aligned}
\frac{-\langle x - x_0, x - x_0\rangle}{4(t_0 - t)} \leq\ & \frac{-\kappa|\hat{x} - \hat{x}_0|^2 + 2(1-\kappa)^{1/2}(t_0 - t)\sup_{[t,t_0]} |\partial_t u(\hat{x}_0, \cdot)||\hat{x} - \hat{x}_0|}{4(t_0 - t)} \\
& + \frac{(t_0 - t)^2 \sup_{[t,t_0]} |\partial_t u(\hat{x}_0, \cdot)|^2}{4(t_0 - t)} \\
\leq\ & -\frac{\kappa}{4(t_0 - a)}|\hat{x} - \hat{x}_0|^2 + \frac{2(1-\kappa)^{1/2}\sup_{[a,t_0]} |\partial_t u(\hat{x}_0, \cdot)|}{4}|\hat{x} - \hat{x}_0| \\
& + \frac{(t_0 - a)\sup_{[a,t_0]} |\partial_t u(\hat{x}_0, \cdot)|^2}{4}.
\end{aligned}$$

We denote the right hand side of this inequality by $Q(|\hat{x} - \hat{x}_0|)$, where the coefficients of the polynomial Q depend on I' and $\hat{x}_0$ but are independent of $t \in I'$.

We would now like an upper bound on the (non-negative) term[15]

$$\begin{aligned}
& -\left\langle H + \frac{(x - x_0)^{\perp}}{2(t_0 - t)}, H + \frac{(x - x_0)^{\perp}}{2(t_0 - t)} \right\rangle \\
=\ & -\langle H, H\rangle - \frac{1}{t_0 - t}\langle H, x - x_0\rangle - \frac{1}{4(t_0 - t)^2}\left\langle (x - x_0)^{\perp}, (x - x_0)^{\perp}\right\rangle \\
\leq\ & |H|^2 + \frac{1}{t_0 - b}|\langle H, x - x_0\rangle| + \frac{1}{4(t_0 - b)^2}|\left\langle (x - x_0)^{\perp}, (x - x_0)^{\perp}\right\rangle|
\end{aligned}$$

[15]We use the fact that $|\langle v, w\rangle| = |v\bar{g}w| \leq |v|.|w|$.

$$
\begin{aligned}
&\leq |H|^2 + \frac{1}{t_0 - b}|H||x - x_0| \\
&\quad + \frac{1}{4(t_0-b)^2}\left| \langle (x-x_0), (x-x_0) \rangle - \left\langle (x-x_0)^\top, (x-x_0)^\top \right\rangle \right| \\
&\leq |H|^2 + \frac{1}{t_0 - b}|H||x - x_0| \\
&\quad + \frac{1}{4(t_0-b)^2}|x - x_0|^2 + \frac{1}{4(t_0-b)^2}\left\langle (x-x_0)^\top, (x-x_0)^\top \right\rangle.
\end{aligned}
$$

We know $|H|$ is uniformly bounded (by our assumptions here), and inequality (3.12) gives

$$
\begin{aligned}
|x - x_0|^2 &= |\hat{x} - \hat{x}_0|^2 + |u(\hat{x}, t) - u(\hat{x}_0, t_0)|^2 \\
&\leq |\hat{x} - \hat{x}_0|^2 + \left((1-\kappa)^{1/2}|\hat{x} - \hat{x}_0| + (t_0 - a)\sup_{[a,t_0]} |\partial_t u(\hat{x}_0, \cdot)| \right)^2.
\end{aligned}
$$

So we just need a bound on

$$
\begin{aligned}
\left| \left\langle (x-x_0)^\top, (x-x_0)^\top \right\rangle \right| &= \langle (x-x_0), \partial_i(\hat{x}, u(\hat{x}, t)) \rangle \, \hat{g}^{ij}(Du) \, \langle (x-x_0), \partial_j(\hat{x}, u(\hat{x}, t)) \rangle \\
&\leq \frac{1}{\kappa}\sum_i \langle x - x_0, (e_i, \partial_i u) \rangle^2 \\
&\leq \frac{1}{\kappa}|x - x_0|^2 \sum_i |(e_i, \partial_i u)|^2,
\end{aligned}
$$

where we have already obtained a bound on $|x - x_0|$ and $|(e_i, \partial_i u)|$ is obviously bounded since $|||Du|||^2 \leq 1 - \kappa$. Combining these inequalities gives

$$
-\left\langle H + \frac{(x-x_0)^\perp}{2(t_0-t)}, H + \frac{(x-x_0)^\perp}{2(t_0-t)} \right\rangle \leq P(|\hat{x} - \hat{x}_0|),
$$

where P is some polynomial with coefficients again independent of $t \in I'$.

Now we recall equation (3.14) and use it to get

$$
\begin{aligned}
&\left| \frac{d}{dt}\int_{\mathcal{M}(t)} \Phi_{X_0}\chi_R - -\int_{\mathcal{M}(t)} \left\langle H + \frac{(x-x_0)^\perp}{2(t_0-t)}, H + \frac{(x-x_0)^\perp}{2(t_0-t)} \right\rangle \Phi_{X_0} \right| \\
= &\left| \int_{\mathcal{M}(t)} \Phi_{X_0}\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)} \right)\chi_R + \left\langle H + \frac{(x-x_0)^\perp}{2(t_0-t)}, H + \frac{(x-x_0)^\perp}{2(t_0-t)} \right\rangle \Phi_{X_0}(1 - \chi_R) \right| \\
\leq &\left| \int_{\mathcal{M}(t)} \Phi_{X_0}\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)} \right)\chi_R \right| \\
&+ \left| \int_{\mathcal{M}(t)} \left\langle H + \frac{(x-x_0)^\perp}{2(t_0-t)}, H + \frac{(x-x_0)^\perp}{2(t_0-t)} \right\rangle \Phi_{X_0}(1 - \chi_R) \right|
\end{aligned}
$$

$$
\begin{aligned}
\leq & \int_{\mathbb{R}^m} \frac{C_0 C}{R^2} \frac{\chi_{B^m_{2R}(0)-B^m_R(0)}}{(4\pi(t_0-b))^{m/2}} \exp[Q(|\hat{x}-\hat{x}_0|)] d\hat{x} \\
& + \int_{\mathbb{R}^m} P(|\hat{x}-\hat{x}_0|) \frac{(1-\chi_{B^m_R(0)})}{(4\pi(t_0-b))^{m/2}} \exp[Q(|\hat{x}-\hat{x}_0|)] d\hat{x},
\end{aligned}
$$

where we have used all of the inequalities above, as well as $\sqrt{\det \hat{g}} \leq 1$. Both integrands in the right hand side are bounded by an integrable function (by the usual Gaussian integral formula, since Q is dominated by the $-|\hat{x}-\hat{x}_0|^2$ term and P is just a polynomial) which is independent of R. Both integrands converge pointwise to zero on $\mathbb{R}^m$ as $R \to \infty$, which allows us to apply the dominated convergence theorem[16] to see that the right hand side of this inequality converges to zero. Since the right hand side is independent of $t \in I'$, this convergence is uniform. So we have

$$
\lim_{R\to\infty} \frac{d}{dt} \int_{\mathcal{M}(t)} \Phi_{X_0} \chi_R = - \int_{\mathcal{M}(t)} \left\langle H + \frac{(x-x_0)^\perp}{2(t_0-t)}, H + \frac{(x-x_0)^\perp}{2(t_0-t)} \right\rangle \Phi_{X_0}.
$$

The uniform convergence allows us to swap the order of the limit and the derivative on the left hand side to get

$$
\begin{aligned}
- \int_{\mathcal{M}(t)} \left\langle H + \frac{(x-x_0)^\perp}{2(t_0-t)}, H + \frac{(x-x_0)^\perp}{2(t_0-t)} \right\rangle \Phi_{X_0} &= \frac{d}{dt} \lim_{R\to\infty} \int_{\mathcal{M}(t)} \Phi_{X_0} \chi_R \\
&= \frac{d}{dt} \int_{\mathcal{M}(t)} \Phi_{X_0},
\end{aligned}
$$

where we have again used a dominated convergence argument (involving Q, etc.) and the fact that χ^m_R converges to 1 pointwise. Since we can do this for any such interval I', the equation above holds for all $t < t_0$. This finally proves the theorem. □

The proof of this theorem should be compared to the proof on page 55 of [5]. Note that the choice of χ_R also gives the possibility of a kind of weighted monotonicity formula (see [5]). We could even weaken the assumption on H, but for now it is enough to assume that it is bounded.

Corollary 3.4.1. *Let $\mathcal{M}$ be as in Theorem 3.4.1, then $\Theta(\mathcal{M}, X, t) \leq 1$ for all $X \in \mathcal{M}$ and all $t < \tau(X)$. Also, $\Theta(\mathcal{M}, X, t) = 1$ for all $X \in \mathcal{M}$ and all $t < \tau(X)$ if and only if $\mathcal{M}$ is a non-moving plane.*

[16] Suppose we are given a sequence of integrable functions on $\mathbb{R}^m$, converging pointwise almost everywhere to some limit function. Suppose that the absolute value of each function in the sequence is bounded by some fixed integrable function. Then the limit of the sequence of integrals of these functions is equal to the integral of the limit function. See Theorem 16.5 of [9].

Proof. Let $Y = (y, s) \in \mathcal{M}$, then we claim that $\lim_{t\to s} \Theta(\mathcal{M}, Y, t) = 1$. We prove this by considering dilations of the flow, using Proposition 3.2.1.

$$\Theta(\mathcal{M}, Y, t) = \Theta(D_{1/(s-t)^{1/2}}(\mathcal{M} - Y), 0, -1) \tag{3.16}$$

and, since the flow is smooth at Y, the flows $D_{1/(s-t)^{1/2}}(\mathcal{M} - Y)$ converge to a non-moving plane as $t \to s$. To understand why, write $\lambda = \sqrt{s-t}$ (which converges to 0 as $t \to s$) and let each of the flows $D_{1/\lambda}(\mathcal{M} - Y)$ be given by the graph of a function u_λ. If the flow $(\mathcal{M} - Y)$ is the graph of a function u, then $u_\lambda(\hat{z}, r) = u(\lambda\hat{z}, \lambda^2 r)/\lambda$ and the definition of the derivative of this function with respect to λ gives us

$$\lim_{\lambda\to 0} u_\lambda(\hat{z}, r) = \lim_{\lambda\to 0} \frac{u(\lambda\hat{z}, \lambda^2 r)}{\lambda} = \partial_\lambda u(\lambda\hat{z}, \lambda^2 r)|_{\lambda=0} = Du(0,0)\cdot\hat{z} + 0\cdot 2r\partial_t u(0,0),$$

and therefore our sequence of flows $D_{1/\lambda}(\mathcal{M} - Y)$ converges pointwise to a non-moving plane as $\lambda \to 0$. We can easily see that $Du_\lambda(\hat{z}, r) = Du(\lambda\hat{z}, \lambda^2 r) \to Du(0,0)$, so that $\det \hat{g}(Du_\lambda)$ converges pointwise to $\det \hat{g}(Du(0,0))$. Also,

$$\sup_{[-1,0]} |\partial_t u_\lambda(0,\cdot)| = \lambda \sup_{[-\lambda^2,0]} |\partial_t u(0,\cdot)| \to 0$$

as $\lambda \to 0$ (since u is smooth). We can use these facts now to apply the dominated convergence theorem to $\Theta(D_{1/\lambda}(\mathcal{M} - Y), 0, -1)$, by again using inequality (3.12) in the usual way to get an upper bound on the exponent of $\Phi_0(\cdot, -1)$ on each of the flows $D_{1/\lambda}(\mathcal{M} - Y)$,

$$\begin{aligned}
& \frac{-|\hat{x} - 0|^2 + |u_\lambda(\hat{x}, -1) - u_\lambda(0,0)|^2}{4(0 - -1)} \\
\leq \quad & \frac{-|\hat{x}|^2 + ((1-\kappa)^{1/2}|\hat{x}| + (0 - -1)\sup_{[-1,0]} |\partial_t u_\lambda(0,\cdot)|)^2}{4} \\
\leq \quad & \frac{-\kappa|\hat{x}|^2 + 2(1-\kappa)^{1/2}|\hat{x}| + 1}{4},
\end{aligned}$$

whenever λ is small enough such that $\sup_{[-1,0]} |\partial_t u_\lambda(0,\cdot)| \leq 1$. Now we have a bound (for all small λ) on the integrands of each $\Theta(D_{1/\lambda}(\mathcal{M} - Y), 0, -1)$ by some function (integrable over $\mathbb{R}^m$), and we know that $D_{1/\lambda}(\mathcal{M} - Y)$ converges pointwise to a non-moving plane. We can therefore apply the dominated convergence theorem to get $\Theta(D_{1/\lambda}(\mathcal{M} - Y), 0, -1) \to 1$ as $\lambda \to 0$, since Θ is always equal to 1 on non-moving planes. Obviously, this fact and equation (3.16) give

$$\Theta(\mathcal{M}, Y, t) \to 1$$

as $t \to s$. The monotonicity theorem tells us that $\Theta(\mathcal{M}, Y, t)$ is non-decreasing with respect to $t < s$ and therefore must be ≤ 1.

For the second part of the corollary, if $\Theta(\mathcal{M}, Y, t) \equiv 1$ then the monotonicity formula gives

$$0 = \frac{d}{dt}\Theta(\mathcal{M}, Y, t) = -\int_{\mathcal{M}(t)} \left\langle H + \frac{(x-y)^\perp}{2(s-t)}, H + \frac{(x-y)^\perp}{2(s-t)} \right\rangle \Phi_Y,$$

and therefore (since normal vectors are timelike or zero)

$$H(x,t) = -(x-y)^\perp / 2(s-t).$$

This means that the flow

$$\mathcal{M}' = (\mathcal{M} - Y) \cap \{X \mid \tau(X) \leq 0\}$$

satisfies equation (3.9) and must be invariant under parabolic dilations. As $\lambda \to \infty$, the flows $D_\lambda \mathcal{M}'$ again converge to a non-moving plane, which must be equal to $\mathcal{M}'$ itself. This is true for all $Y \in \mathcal{M}$, so $\mathcal{M}$ must be a non-moving plane. □

3.5 Local Monotonicity

In this section we prove a kind of local monotonicity theorem which will be used to prove a local regularity theorem later. We continue to make Assumptions 1, 2 and 3, but we add:

Assumption 5: With $\mathcal{M}$ as in Assumption 2, $\Omega \times I$ is bounded and u is continuous on its closure.

Note that if our flow (satisfying Assumptions 1, 2, 3) is smooth on an interval $I = [a, b)$, then Proposition 3.3.3 tells us that it can be extended continuously to time b. By taking a subset of Ω if necessary (remember that we are interested in local theorems here), Assumption 5 will hold. We also note that inequality (3.11) will continue to hold on the closure of $\Omega \times I$ when Assumption 5 holds.

We can choose a C^2 function $\phi : \mathbb{R}^m \to \mathbb{R}$ which satisfies

$$\chi_{B^m_{1/2}(0)} \leq \phi \leq \chi_{B^m_1(0)} \quad \text{and} \quad |D^2\phi| \leq C_1,$$

where C_1 is some positive constant depending only on m. Then, for any spacetime point $X_0 = (\hat{x}_0, \tilde{x}_0, t_0)$ and any $\rho > 0$, we define a function on $\mathbb{R}^{m+n}$ by

$$\phi_{\rho,X_0}(x) = \phi_{\rho,X_0}(\hat{x}, \tilde{x}) = \phi\left(\frac{\hat{x} - \hat{x}_0}{\rho}\right),$$

which will have $\chi_{B^m_{\rho/2}(\hat{x}_0)\times\mathbb{R}^n} \leq \phi_{\rho,X_0} \leq \chi_{B^m_\rho(\hat{x}_0)\times\mathbb{R}^n}$ and $|D^2\phi_{\rho,X_0}| \leq C_1/\rho^2$.

It will also be convenient now for us to define the sets $Q^{m,n,1}_\rho(X) = B^m_\rho(\hat{x}) \times \mathbb{R}^n \times (t - \rho^2, t)$ and $P^{m,n,1}_\rho(X) = B^m_\rho(\hat{x}) \times \mathbb{R}^n \times (t - \rho^2, t + \rho^2)$ for any spacetime point $X = (\hat{x}, \tilde{x}, t)$.

Definition 3.5.1. *Let $\mathcal{M}$ be a flow satisfying Assumption 2. If $X_0 \in \mathbb{R}^{m+n,1}$ and $\rho > 0$ are such that $Q^{m,n,1}_\rho(X_0) \subset \Omega \times \mathbb{R}^n \times I$, then we define*

$$\Theta(\mathcal{M}, X_0, t, \rho) = \int_{x\in\mathcal{M}(t)} \Phi_{X_0}(x,t)\phi_{\rho,X_0}(x)$$

for $t < \tau(X_0)$ in I.

With Θ as in this definition, we have the following two simple but useful facts.

Proposition 3.5.1. $\Theta(D_\lambda(\mathcal{M} - X), Y, t, \rho) = \Theta(\mathcal{M}, X + D_{1/\lambda}Y, \tau(X) + t/\lambda^2, \rho/\lambda)$.

Proof. This is proved exactly as in the proof of Proposition 3.2.1 by using the transformation formula for integrals. □

Proposition 3.5.2. $\Theta(\mathcal{M}, X, s, \rho)$ *is continuous with respect to $X \in \mathcal{M}$.*

Proof. Remember that we are considering smooth flows satisfying Assumption 2, so take a sequence $X_J = (\hat{x}_J, u(\hat{x}_J, t_J), t_J)$ on $\mathcal{M}$ which converges to $X = (\hat{x}, u(\hat{x}, t), t)$ as $J \to \infty$. Then $\Theta(\mathcal{M}, X_J, s, \rho)$ is the integral of

$$\frac{\exp\left(\frac{-|\hat{z}-\hat{x}_J|^2+|u(\hat{z},s)-u(\hat{x}_J,t_J)|^2}{4(t_J-s)}\right)}{(4\pi(t_J - s))^{m/2}}\phi\left(\frac{\hat{z} - \hat{x}_J}{\rho}\right)\sqrt{\det \hat{g}(Du(\hat{z}, s))} \tag{3.17}$$

over $\hat{z} \in \Omega$, and this function obviously converges pointwise to the integrand in $\Theta(\mathcal{M}, X, s, \rho)$. But $X_J \to X$ and $t > s$ imply that we can take some small $R > 0$ such that $\hat{x}_J \in B^m_R(\hat{x})$ and $0 < t - s - R^2 < t_J - s < t - s + R^2$ for large enough J. Then, by smoothness of u, we have a bound $\sup_{(s,t_J)} |\partial_t u(\hat{x}_J, \cdot)| \leq \sup_{B^m_R(\hat{x})} \sup_{(s,t+R^2)} |\partial_t u|$ independent of J. Using this with inequality (3.12) in the usual way, along with $\phi \leq 1$ and $\sqrt{\det \hat{g}} \leq 1$, we get a bound on (3.17) by some integrable function independent of J. This allows us to apply dominated convergence theorem to get $\Theta(\mathcal{M}, X_J, s, \rho) \to \Theta(\mathcal{M}, X, s, \rho)$. $\square$

Now we prove a kind of local monotonicity theorem.

Theorem 3.5.1. *Let $\mathcal{M}$ satisfy Assumptions 1, 2, 3, 5, and let $\rho > 0$. Then there exist positive constants $C_2(\mathcal{M}, \rho)$ and $\delta(\mathcal{M}, \rho) < \rho^2$ such that, whenever $X_0 \in \bar{\mathcal{M}}$ is such that $Q^{m,n,1}_\rho(X_0)$ is contained in $\Omega \times \mathbb{R}^n \times I$, the function*

$$t \mapsto \Theta(\mathcal{M}, X_0, t, \rho) + C_2 t$$

will be non-decreasing with respect to $t \in (\tau(X_0) - \delta, \tau(X_0))$.

Note that C_2 and δ will be independent of such points X_0.

Proof. We know from equation (3.7) that (since our assumptions here imply that ϕ_{ρ,X_0} has compact support on each $\mathcal{M}(t)$)

$$\frac{d}{dt}\Theta(\mathcal{M}, X_0, t, \rho) \geq \int_{\mathcal{M}(t)} \Phi_{X_0} \left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)} \right) \phi_{\rho,X_0}. \tag{3.18}$$

As before (see the proof of Theorem 3.4.1), it is easy enough to show that

$$\left| \left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)} \right) \phi_{\rho,X_0} \right| \leq C_3 \chi_{B^m_\rho(\hat{x}_0) \times \mathbb{R}^n - B^m_{\rho/2}(\hat{x}_0) \times \mathbb{R}^n},$$

where $C_3 = C_3(\kappa, \rho)$ is constant. Let $\hat{x}, \hat{y} \in \bar{\Omega}$ and $t < s$ in $\bar{I}$ be such that $\rho/2 < |\hat{x} - \hat{y}| < \rho$. Then, by (3.11) and the triangle inequality, we have

$$\begin{aligned} -|\hat{x} - \hat{y}|^2 + |u(\hat{x}, s) - u(\hat{y}, t)|^2 \;\leq\; & -|\hat{x} - \hat{y}|^2 \\ & + (|u(\hat{x}, s) - u(\hat{x}, t)| + |u(\hat{x}, t) - u(\hat{y}, t)|)^2 \end{aligned}$$

$$
\begin{aligned}
&\le -|\hat{x}-\hat{y}|^2 \\
&\quad +(|u(\hat{x},s)-u(\hat{x},t)|+(1-\kappa)^{1/2}|\hat{x}-\hat{y}|)^2 \\
&\le -\kappa|\hat{x}-\hat{y}|^2 \\
&\quad +2(1-\kappa)^{1/2}|\hat{x}-\hat{y}||u(\hat{x},s)-u(\hat{x},t)| \\
&\quad +|u(\hat{x},s)-u(\hat{x},t)|^2 \\
&\le -\kappa\rho^2/4 \\
&\quad +2(1-\kappa)^{1/2}\rho|u(\hat{x},s)-u(\hat{x},t)| \\
&\quad +|u(\hat{x},s)-u(\hat{x},t)|^2. \qquad (3.19)
\end{aligned}
$$

But, by uniform continuity of u (since it is continuous on the closure of $\Omega \times I$), we can take $\delta(\mathcal{M},\rho)$ (not depending on $\hat{x},\hat{y},s,t$) such that $|u(\hat{x},t)-u(\hat{x},s)|$ will be small enough that the right hand side of the inequality above will be $\le -\kappa\rho^2/8$ whenever $|t-s|<\delta$.[17] Taking $s=\tau(X_0)$ and combining the above inequalities with the fact that $\sqrt{\det \hat{g}} \le 1$ gives

$$
\left|\int_{\mathcal{M}(t)} \Phi_{X_0}\left(\frac{d}{dt}-\Delta_{\mathcal{M}(t)}\right)\phi_{\rho,X_0}\right| \le \int_\Omega \frac{C_3\chi_{B^m_\rho(\hat{x}_0)-B^m_{\rho/2}(\hat{x}_0)}}{(4\pi(\tau(X_0)-t))^{m/2}}\exp\left(\frac{-\rho^2\kappa/8}{4(\tau(X_0)-t)}\right),
$$

for $0<\tau(X_0)-t<\delta$. Taking $t\to\tau(X_0)$ in the right hand side shows that it is bounded by some finite constant $C_4(\mathcal{M},\rho)$ for these values of t. Therefore $\frac{d}{dt}\Theta(\mathcal{M},X_0,t,\rho) \ge -C_4$ for $t\in(\tau(X_0)-\delta,\tau(X_0))$ and this proves the theorem. □

Corollary 3.5.1. *Let $\mathcal{M}$ be as in Theorem 3.5.1. If X_0 lies in the closure $\bar{\mathcal{M}}$ and $\rho_0>0$ is such that $Q^{m,n,1}_{\rho_0}(Y)\subset\Omega\times\mathbb{R}^n\times I$ for all $Y\in Q^{m,n,1}_{\rho_0}(X_0)$, and if*

$$
\lim_{t\to\tau(X_0)} \Theta(\mathcal{M},X_0,t,\rho_0) > 1-\epsilon
$$

for some $\epsilon>0$, then there exists $\rho\in(0,\rho_0)$ such that

$$
\Theta(\mathcal{M},Y,t,\rho_0) \ge 1-\epsilon
$$

for all $Y\in Q^{m,n,1}_\rho(X_0)\cap\mathcal{M}$ and all $t\in(\tau(Y)-\rho^2,\tau(Y))$.

[17] Uniform continuity implies that for any $\epsilon>0$ there exists $\delta>0$ such that $||(\hat{x},s)-(\hat{y},t)||<\delta$ implies $|u(\hat{x},s)-u(\hat{y},t)|<\epsilon$. Taking $\hat{x}=\hat{y}$ and a small enough ϵ here proves our claim.

Proof. Let $\lim_{t\to\tau(X_0)}\Theta(\mathcal{M},X_0,t,\rho_0)\geq 1-\epsilon+\eta$ for some $\eta>0$ (note that this limit exists in $\mathbb{R}\cup\{\infty\}$ by the local monotonicity theorem). Then there must exist $\rho_1\in(0,\rho_0]$ such that

$$\Theta(\mathcal{M},X_0,\tau(X_0)-\rho_1^2,\rho_0)>1-\epsilon+\eta/2.$$

We can choose ρ_1 to be as small as we like, so we take $\rho_1^2<\min\{\delta(\mathcal{M},\rho_0),\eta/4C_2(\mathcal{M},\rho_0)\}$ (with δ and C_2 as in the Theorem 3.5.1). By continuity, there will exist $\rho\in(0,\rho_1)$ such that, for all $Y\in Q_\rho^{m,n,1}(X_0)\cap\mathcal{M}$,

$$\Theta(\mathcal{M},Y,\tau(X_0)-\rho_1^2,\rho_0)>1-\epsilon+\eta/4$$

and $(\tau(Y)-\rho^2,\tau(Y))\subset(\tau(X_0)-\rho_1^2,\tau(X_0))\subset(\tau(X_0)-\delta,\tau(X_0))$. So we can apply Theorem 3.5.1 to $\Theta(\mathcal{M},Y,t,\rho_0)$ for $t\in(\tau(Y)-\rho^2,\tau(Y))$ to get

$$\Theta(\mathcal{M},Y,\tau(X_0)-\rho_1^2,\rho_0)+C_2(\tau(X_0)-\rho_1^2)\leq\Theta(\mathcal{M},Y,t,\rho_0)+C_2t,$$

which implies

$$\begin{aligned}\Theta(\mathcal{M},Y,t,\rho_0) &\geq C_2(\tau(X_0)-t-\rho_1^2)+1-\epsilon+\eta/4\\ &\geq 1-\epsilon+(\eta/4-\rho_1^2C_2),\end{aligned}$$

for all such Y and t, where the last term is non-negative by our choice of ρ_1. $\square$

Proposition 3.5.3. *Let $\mathcal{M}$ satisfy Assumptions 2, 3 and 5, and let X_0 and ρ be as in Theorem 3.5.1, then*

$$\lim_{t\to\tau(X_0)}\Theta(\mathcal{M},X_0,t,\rho)=\lim_{t\to\tau(X_0)}\Theta(\mathcal{M},X_0,t).$$

In particular, the limit on the left hand side is independent of ρ.

Proof. It is easy to see that, if we write $X_0=(\hat{x}_0,u(\hat{x}_0,t_0),t_0)$,

$$\begin{aligned}0 &\leq \Theta(\mathcal{M},X_0,t)-\Theta(\mathcal{M},X_0,t,\rho)\\ &= \int_{\mathcal{M}(t)}\Phi_{X_0}(1-\phi_{\rho,X_0})\\ &= \int_\Omega\frac{\exp\left(\frac{-|\hat{x}-\hat{x}_0|^2+|u(\hat{x},t)-u(\hat{x}_0,t_0)|^2}{4(t_0-t)}\right)}{(4\pi(t_0-t))^{m/2}}\left(1-\phi\left(\frac{\hat{x}-\hat{x}_0}{\rho}\right)\right)\sqrt{\det\hat{g}}d\hat{x}. \quad (3.20)\end{aligned}$$

But $\sqrt{\det \hat{g}} < 1$ and $1 - \phi\left(\frac{\hat{x}-\hat{x}_0}{\rho}\right) \leq 1$ is zero for $\hat{x} \in B^m_{\rho/2}(\hat{x}_0)$. Therefore we only need to consider $|\hat{x} - \hat{x}_0| \geq \rho/2$ and, as in inequality (3.19), we get

$$\begin{aligned}
& -|\hat{x} - \hat{x}_0|^2 + |u(\hat{x}, t) - u(\hat{x}_0, t_0)|^2 \\
\leq \;& -\kappa|\hat{x} - \hat{x}_0| + 2(1-\kappa)^{1/2}|\hat{x} - \hat{x}_0||u(\hat{x}_0, t) - u(\hat{x}_0, t_0)| + |u(\hat{x}_0, t) - u(\hat{x}_0, t_0)|^2 \\
\leq \;& -\kappa\rho^2/4 + 2(1-\kappa)^{1/2}\mathrm{diam}\Omega|u(\hat{x}_0, t) - u(\hat{x}_0, t_0)| + |u(\hat{x}_0, t) - u(\hat{x}_0, t_0)|^2 \\
\leq \;& -\kappa\rho^2/8,
\end{aligned}$$

where the last step again involves choosing $|u(\hat{x}_0, t) - u(\hat{x}_0, t_0)|$ small enough (by continuity) by taking t close enough to t_0. Therefore, for such t, the right hand side of inequality (3.20) is less than or equal to

$$\int_\Omega \frac{\exp\left((-\kappa\rho^2/8)/4(t_0 - t)\right)}{(4\pi(t_0 - t))^{m/2}} d\hat{x},$$

which converges to 0 as $t \to t_0$. □

3.6 Local Regularity

In [24], a regularity theorem for mean curvature flows in Euclidean spaces is proved. To do this, a kind of local $C^{2,\alpha}$ norm is used (defined at each point of a flow and denoted by $K_{2,\alpha}$). If, for a sequence of $C^{2,\alpha}$ flows $\mathcal{M}_J$, this norm is uniformly bounded on compact subsets as $J \to \infty$, then a version of the Arzela-Ascoli theorem (Theorem 2.6 of [24]) gives local parabolic C^2 convergence of a subsequence to some locally $C^{2,\alpha}$ flow. However, the definition of this norm involves rotations, which would cause problems in the semi-Euclidean case (for example, because of the spacelike condition). It is convenient for us to define a slightly different quantity with similar properties. The idea will be to use the gradient bound (from the spacelike assumption) to ignore the first few terms in the $C^{2,\alpha}$ norm, thus removing the need to translate and rotate in the definition of $K_{2,\alpha}$.

Definition 3.6.1. *Suppose that we have a spacelike flow $\mathcal{M}$ satisfying Assumption 2 (not necessarily a mean curvature flow) and that $X \in \Omega \times \mathbb{R}^n \times I$. Then, for any constant $\alpha \in (0, 1)$, we define $G_{2,\alpha}(\mathcal{M}, X)$ to be the infimum of the numbers $\lambda > 0$*

such that

$$u_{\lambda,X}|_{U^{m,1}} \textit{ satisfies } [Du_{\lambda,X}]_\alpha + ||D^2 u_{\lambda,X}||_{0,\alpha} + ||\partial_t u_{\lambda,X}||_{0,\alpha} \leq 1, \tag{3.21}$$

where $u_{\lambda,X}$ *is the function whose graph gives the flow* $D_\lambda(\mathcal{M} - X)$*, and* $u_{\lambda,X}|_{U^{m,1}}$ *is the restriction to the intersection of its domain with* $U^{m,1} = B_1^m(0) \times (-1, 0]$.

This quantity will be finite when the flow is smooth (to understand why, see how each term in (3.21) is affected by dilations[18]). It is important to note that, for any $X = (\hat{x}, \tilde{x}, t)$, $G_{2,\alpha}(\mathcal{M}, X)$ is independent of $\tilde{x}$ (since the definition only involves derivatives of u). We will also need the obvious facts that this quantity will be zero on non-moving planes and that we have $G_{2,\alpha}(D_\lambda(\mathcal{M} - X), 0) = G_{2,\alpha}(\mathcal{M}, X)/\lambda$.

The most important property of $G_{2,\alpha}$ is a version of the Arzela-Ascoli theorem. Roughly, if we have a sequence of smooth spacelike flows $\mathcal{M}_J$, each containing the origin and with $G_{2,\alpha}(\mathcal{M}_J, \cdot)$ uniformly bounded on compact subsets of spacetime as $J \to \infty$, then we have local parabolic C^2 convergence of some subsequence to a locally $C^{2,\alpha}$ limit flow. Comparing $G_{2,\alpha}$ to $K_{2,\alpha}$ and applying Theorem 2.6 of [24] gives us this fact, but we will still explain in detail in Proposition 3.7.1 in a special case. Furthermore, if each of the flows satisfies the system in Proposition 3.3.1 then so will the limit (by the C^2 convergence), which must then be smooth (again see the proof of Proposition 3.7.1).

The next theorem is the most important result of this chapter. It is a version of White's local regularity theorem. The proof should be compared to a combination of the proofs of Theorem 3.1 of [24] and Theorem 5.6 of [5]. As in [5], we will use the local version of Θ so that we can avoid the boundary of spatial slices the flow. As in [24], we aim for bounds on the $C^{2,\alpha}$ norm and use the Schauder estimates, rather than aiming for bounds on the second fundamental form and using related interior estimates as in [5].

[18]For example, $|D^2 u_{\lambda,X}(\cdot,\cdot)| = |D^2 u(\cdot/\lambda + \hat{x}, \cdot/\lambda^2 + \tau(X))|/\lambda$ which, for large enough λ, will be less than any $\epsilon > 0$ on $U^{m,1}$ (since u is C^∞). Similar reasoning applies to all other terms in the definition of $G_{2,\alpha}$, allowing us to take λ large enough such that the sum of these will be ≤ 1.

Theorem 3.6.1. *Let $\alpha, \kappa \in (0,1)$ be given. Then there exist positive constants ϵ and C_5 such that if*

(a) $\mathcal{M}$ is any mean curvature flow satisfying Assumptions 1, 2 and 3, with $\sup I = 0 \in I$ and with $u(0,0) = 0$,

(b) $\rho_0 > 1$ is such that $Q^{m,n,1}_{\rho_0}(Y)$ is contained in $\Omega \times \mathbb{R}^n \times I$ and

$$\Theta(\mathcal{M}, Y, t, \rho_0) \geq 1 - \epsilon$$

for all $Y \in Q^{m,n,1}_1(0) \cap \mathcal{M}$ and all $t \in (\tau(Y) - 1, \tau(Y))$,

then

$$\sup_{X \in Q^{m,n,1}_1(0)} G_{2,\alpha}(\mathcal{M}, X) d(X, P^{m,n,1}_1(0)) \leq C_5.$$

It is important to notice that the constants ϵ and C_5 will depend on κ, α, m, n, but will be independent of $\mathcal{M}$. It is also worth noting that, since $G_{2,\alpha}$ scales like the reciprocal of parabolic distance, the inequality in the conclusion of the theorem is invariant under parabolic dilations.

Proof. Let $\bar{\epsilon}$ be the infimum of numbers $\epsilon > 0$ for which the theorem fails (i.e. for which no such C_5 exists). We need $\bar{\epsilon} > 0$, so we assume $\bar{\epsilon} = 0$ to get a contradiction. We take a sequence $\epsilon_J \to \bar{\epsilon}$ with $\epsilon_J > \bar{\epsilon}$. Then there exist sequences $\mathcal{M}_J$ and $\rho_J > 1$, satisfying all of the assumptions of the theorem (with the same α and κ), but with $\epsilon_J, \mathcal{M}_J, \rho_J$ in place of $\epsilon, \mathcal{M}, \rho_0$ (respectively), and with

$$\gamma_J = \sup_{X \in Q^{m,n,1}_1(0)} d(X, P^{m,n,1}_1(0)) G_{2,\alpha}(\mathcal{M}_J, X) \to \infty$$

as $J \to \infty$. Each γ_J is finite, since $\mathcal{M}_J$ is smooth and the closure of the projection of $Q^{m,n,1}_1(0)$ onto the spacetime $\mathbb{R}^{m,1}$ is a compact subset of the set $\Omega_J \times I_J$ corresponding to $\mathcal{M}_J$. For each J we can choose $Y_J \in Q^{m,n,1}_1(0)$ such that

$$G_{2,\alpha}(\mathcal{M}_J, Y_J) d(Y_J, P^{m,n,1}_1(0)) \geq \frac{\gamma_J}{2},$$

and we can assume that $Y_J \in \mathcal{M}_J$.[19] We define $\lambda_J = G_{2,\alpha}(\mathcal{M}_J, Y_J)$ and consider

[19] Remember that $G_{2,\alpha}(\mathcal{M}, (\hat{x}, \tilde{x}, t))$ is independent of $\tilde{x}$, and so is $d((\hat{x}, \tilde{x}, t), P^{m,n,1}_1(0))$.

the flows[20]

$$\tilde{\mathcal{M}}_J = D_{\lambda_J}(\mathcal{M}_J - Y_J),$$

which all contain the origin (in spacetime). Then $G_{2,\alpha}(\tilde{\mathcal{M}}_J, 0) = 1$ for all J, and $D_{\lambda_J}(P_1^{m,n,1}(0) - Y_J) = P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)$. But now

$$\begin{aligned} \frac{\gamma_J}{2} &\leq G_{2,\alpha}(\mathcal{M}_J, Y_J)d(Y_J, P_1^{m,n,1}(0)) \\ &= G_{2,\alpha}(\tilde{\mathcal{M}}_J, 0)d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)) \\ &= d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)), \end{aligned}$$

so $d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)) \to \infty$ since $\gamma_J \to \infty$ as $J \to \infty$. Let X be a point in $Q_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)$, then

$$d(X, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J))G_{2,\alpha}(\tilde{\mathcal{M}}_J, X) \leq \gamma_J \leq 2d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)),$$

and therefore

$$G_{2,\alpha}(\tilde{\mathcal{M}}_J, X) \leq \frac{2d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J))}{d(X, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J))}.$$

The triangle inequality gives $||0 - Y|| \leq ||0 - X|| + ||Y - X||$, and taking the supremum over all $Y \notin P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)$ gives

$$\begin{aligned} d(X, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)) &\geq d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)) - ||X|| \\ \Rightarrow G_{2,\alpha}(\tilde{\mathcal{M}}_J, X) &\leq \frac{2d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J))}{d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)) - ||X||} \\ &= \frac{2}{1 - ||X||/d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J))}, \end{aligned} \tag{3.22}$$

whenever the right hand side is positive. Since $d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)) \to \infty$, this inequality tells us that $G_{2,\alpha}(\tilde{\mathcal{M}}_J, X)$ is uniformly bounded (as $J \to \infty$) on compact subsets of spacetime with $\tau(X) \leq 0$.[21] This allows us to apply Proposition 3.7.1 to the sequence $\tilde{\mathcal{M}}_J \cap \{X \mid \tau(X) \leq 0\}$ to get parabolic C^2 convergence, on compact subsets of $\mathbb{R}^m \times (-\infty, 0]$, of a subsequence to a limit flow $\mathcal{M}'$. We can

[20] Note that the flows $\mathcal{M}_J$ and $\tilde{\mathcal{M}}_J$ will be graphs of functions u_J and $\tilde{u}_J$ on sets $\Omega_J \times I_J$ and $\tilde{\Omega}_J \times \tilde{I}_J$ respectively, where $\sup I_J = 0 \Rightarrow \sup \tilde{I}_J = \tau(-D_{\lambda_J}Y_J) > 0$.

[21] For example, for any such compact set we can assume $G_{2,\alpha}(\tilde{\mathcal{M}}_J, X) \leq 4$ for all X in this set by assuming $||X|| \leq R$ (by compactness) and taking J large such that $d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J}Y_J)) \geq 2R$.

assume that this subsequence is our original sequence, and will therefore continue to use the notation $\tilde{\mathcal{M}}_J$. The limit $\mathcal{M}'$ will be a smooth entire graph defined on $\mathbb{R}^m \times (-\infty, 0]$ (since $\lambda_J \to \infty$). It will be the graph of a function u' satisfying the system in Proposition 3.3.1 (since the convergence is C^2) and will therefore be a mean curvature flow for times < 0. Also, since the gradient bound is unaffected by parabolic dilations, $\sup |||Du'|||^2 \leq 1 - \kappa$. Proposition 3.7.1 tells us that $\mathcal{M}'$ has uniformly bounded mean curvature. This allows us to apply the monotonicity theorem and related results to the flow.

Now we use the assumption that $\Theta(\mathcal{M}_J, Y, s, \rho_J) \geq 1 - \epsilon_J$ for all $Y \in Q_1^{m,n,1}(0) \cap \mathcal{M}_J$, $s \in (\tau(Y) - 1, \tau(Y))$. By Proposition 3.5.1, this is equivalent to the inequality $\Theta(\tilde{\mathcal{M}}_J, Y, s, \lambda_J \rho_J) \geq 1 - \epsilon_J$ for $Y \in Q_{\lambda_J}^{m,n,1}(-D_{\lambda_J} Y_J) \cap \tilde{\mathcal{M}}_J$ and $s \in (\tau(Y) - \lambda_J^2, \tau(Y))$. Given any $Z = (\hat{z}, u'(\hat{z}, t), t) \in \mathcal{M}'$ with $s < t < 0$, we can take a sequence $Z_J = (\hat{z}, \tilde{u}_J(\hat{z}, t), t) \in \tilde{\mathcal{M}}_J$ with $Z_J \to Z$. Then, for large enough J, the fact that $d(0, P_{\lambda_J}^{m,n,1}(-D_{\lambda_J} Y_J)) \to \infty$ implies that Z_J (which is bounded since it converges) will be in $Q_{\lambda_J}^{m,n,1}(-D_{\lambda_J} Y_J)$. Obviously we will have $s \in (\tau(Z_J) - \lambda_J^2, \tau(Z_J))$ for all large J. This gives $\Theta(\tilde{\mathcal{M}}_J, Z_J, s, \lambda_J \rho_J) \geq 1 - \epsilon_J$, and we want to apply the dominated convergence theorem to this inequality. We see easily that $\Theta(\tilde{\mathcal{M}}_J, Z_J, s, \lambda_J \rho_J)$ is equal to

$$\int_{\tilde{\Omega}_J} \frac{\exp\left(\frac{-|\hat{x}-\hat{z}|^2 + |\tilde{u}_J(\hat{x},s) - \tilde{u}_J(\hat{z},t)|^2}{4(t-s)}\right)}{(4\pi(t-s))^{m/2}} \phi\left(\frac{\hat{x} - \hat{z}}{\lambda_J \rho_J}\right) \sqrt{\det \hat{g}(D\tilde{u}_J(\hat{x}, s))} d\hat{x}, \tag{3.23}$$

where we can think of each of these integrals as an integral over $\mathbb{R}^m$ since ϕ is compactly supported in $\tilde{\Omega}_J$. By the C^2 convergence $\tilde{u}_J \to u'$ and the fact that $\rho_J \lambda_J \to \infty$ with $\phi \equiv 1$ in some ball with centre 0, the integrands above will converge pointwise to the integrand

$$\frac{\exp\left(\frac{-|\hat{x}-\hat{z}|^2 + |u'(\hat{x},s) - u'(\hat{z},t)|^2}{4(t-s)}\right)}{(4\pi(t-s))^{m/2}} \times 1 \times \sqrt{\det \hat{g}(Du'(\hat{x}, s))}$$

of the integral $\Theta(\mathcal{M}', Z, s)$. But we have $\phi \leq 1$, $\sqrt{\det \hat{g}} \leq 1$ and $t - s > 0$ independent of J, as well as

$$\begin{aligned}-|\hat{x}-\hat{z}|^2+|\tilde{u}_J(\hat{x},s)-\tilde{u}_J(\hat{z},t)|^2 \quad\leq\quad & -\kappa|\hat{x}-\hat{z}|^2 \\ & +2(1-\kappa)^{1/2}|\hat{x}-\hat{z}|(t-s)\sup_{(s,t)}|\partial_t\tilde{u}_J(\hat{z},\cdot)| \\ & +(t-s)^2\sup_{(s,t)}|\partial_t\tilde{u}_J(\hat{z},\cdot)|^2,\end{aligned}$$

by inequality (3.12). By the parabolic C^2 convergence, we can assume for large J that $\sup_{(s,t)}|\partial_t\tilde{u}_J(\hat{z},\cdot)|$ is arbitrarily close to $\sup_{(s,t)}|\partial_t u'(\hat{z},\cdot)|$, which is finite (by smoothness of u') and independent of J. These inequalities combine to give a bound on the integrands of (3.23) by some function, independent of J and integrable over $\mathbb{R}^m$. This allows us to apply the dominated convergence theorem to get

$$\Theta(\mathcal{M}',Z,s)\leftarrow\Theta(\tilde{\mathcal{M}}_J,Z_J,s,\lambda_J\rho_J)\geq 1-\epsilon_J\to 1-\bar{\epsilon}.$$

So, for all $Z\in\mathcal{M}'$ with $s<\tau(Z)<0$, we have $\Theta(\mathcal{M}',Z,s)\geq 1-\bar{\epsilon}$.

Now we assume that $\bar{\epsilon}=0$. Since $\mathcal{M}'$ is entire, the fact that $\Theta(\mathcal{M}',Z,s)\geq 1$ (whenever $s<\tau(Z)<0$) implies by Corollary 3.4.1 (since H is bounded) that $\Theta(\mathcal{M}',Z,s)\equiv 1$ and therefore $\mathcal{M}'$ must be a non-moving plane for times <0 (and then for times ≤ 0 by smoothness).

Let u' be as above and consider the constant coefficient linear parabolic operator $\partial_t-\hat{g}^{ij}(Du')\partial_{ij}$ applied to the functions $\tilde{u}_J$. Proposition 3.3.1 then gives

$$\left(\partial_t-\hat{g}^{ij}(Du')\partial_{ij}\right)\tilde{u}_J=\left(\hat{g}^{ij}(D\tilde{u}_J)-\hat{g}^{ij}(Du')\right)\partial_{ij}\tilde{u}_J.$$

Since u' is linear and independent of time, $\partial_{ij}u'=\partial_t u'=0$ and therefore

$$\begin{aligned}\left(\partial_t-\hat{g}^{ij}(Du')\partial_{ij}\right)(\tilde{u}_J-u') &= \left(\partial_t-\hat{g}^{ij}(Du')\partial_{ij}\right)\tilde{u}_J \\ &= (\hat{g}^{ij}(D\tilde{u}_J)-\hat{g}^{ij}(Du'))\partial_{ij}\tilde{u}_J.\end{aligned}$$

For the subset $U_2^{m,1}(0)$ of $\mathbb{R}^m\times(-\infty,0]$, the Schauder estimates for linear parabolic equations (see Theorem B.3.2) give us

$$\begin{aligned}||(\tilde{u}_J - u')|_{U_2^{m,1}(0)}||_{2,\alpha} &\leq C_6|| \left(\partial_t - \hat{g}^{ij}(Du')\partial_{ij}\right)(\tilde{u}_J - u')|_{U_4^{m,1}(0)}||_{0,\alpha} \\ &\quad + C_6 \sup_{U_4^{m,1}(0)} |\tilde{u}_J - u'| \\ &= C_6|| \left(\hat{g}^{ij}(D\tilde{u}_J) - \hat{g}^{ij}(Du')\right)\partial_{ij}\tilde{u}_J|_{U_4^{m,1}(0)}||_{0,\alpha} \\ &\quad + C_6 \sup_{U_4^{m,1}(0)} |\tilde{u}_J - u'|,\end{aligned}$$

whenever J is large enough that $U_6^{m,1}(0) \subset \tilde{\Omega}_J \times \tilde{I}_J$, and where the constant C_6 will depend on m, n, α, κ. But both terms on the right hand side converge to 0 as $J \to \infty$ by the fact that $\partial_{ij}\tilde{u}_J$ is bounded in $C^{0,\alpha}$ on compact subsets (by inequality (3.22)) and the fact that

$$\left(\hat{g}^{ij}(D\tilde{u}_J) - \hat{g}^{ij}(Du')\right) \to 0$$

in C^1 on compact sets (since $\hat{g}^{ij}(p)$ depends smoothly on p when $|||p||| < 1$ and since convergence of $\tilde{u}_J$ to u' is C^2 on compact sets). This means that, on $U_2^{m,1}(0)$, the convergence $\tilde{u}_J \to u'$ is $C^{2,\alpha}$. In particular, the terms of the $C^{2,\alpha}$ norm of $\tilde{u}_J$ involved in the definition of $G_{2,\alpha}(\tilde{\mathcal{M}}_J, 0)$ will converge to 0 (since these terms are zero on u'). This finally gives a contradiction because we dilated in such a way that $G_{2,\alpha}(\tilde{\mathcal{M}}_J, 0) = 1 \geq 1/2$ for every J, which implies that $[D\tilde{u}_J]_\alpha + ||D^2\tilde{u}_J||_{0,\alpha} + ||\partial_t\tilde{u}_J||_{0,\alpha}$ is bounded from below, independently of J, on the set $U_{1/(1/2)}^{m,1}(0) \cap \tilde{\Omega} \times \tilde{I} = U_2^{m,1}(0)$ (see inequality (3.32)). This contradiction means that $\bar{\epsilon}$ cannot be zero. $\square$

Corollary 3.6.1. *Let ϵ and C_5 be as in Theorem 3.6.1. Suppose that $\mathcal{M}$ is a mean curvature flow satisfying Assumptions 1, 2 and 3, with $X_0 \in \mathcal{M}$ and $\tau(X_0) = \sup I$.*[22] *Suppose that $\rho_0 > \rho > 0$ are such that $Q_{\rho_0}^{m,n,1}(Y)$ is contained in $\Omega \times \mathbb{R}^n \times I$ and*

$$\Theta(\mathcal{M}, Y, s, \rho_0) \geq 1 - \epsilon$$

for all $Y \in Q_\rho^{m,n,1}(X_0) \cap \mathcal{M}$ and all $s \in (\tau(Y) - \rho^2, \tau(Y))$. Then

$$\sup_{\mathcal{M} \cap Q_\rho^{m,n,1}(X_0)} G_{2,\alpha}(\mathcal{M}, \cdot)d(\cdot, P_\rho^{m,n,1}(X_0)) \leq C_5.$$

[22] By these assumptions, the flow will be smooth at time $\tau(X_0)$, since we are taking X_0 to be a point *on* the flow.

Proof. We take the dilated and translated flow $\mathcal{M}' = D_{1/\rho}(\mathcal{M} - X_0)$. Then, by Proposition 3.5.1 and our assumptions in this corollary, we have

$$\Theta(\mathcal{M}', Z, r, \rho_0/\rho) = \Theta(\mathcal{M}, X_0 + D_\rho Z, \tau(X_0) + \rho^2 r, \rho_0) \geq 1 - \epsilon$$

(where $\rho_0/\rho > 1$) when $D_\rho Z + X_0 \in Q_\rho^{m,n,1}(X_0) \cap \mathcal{M}$ and $\rho^2 r + \tau(X_0) \in (\tau(X_0 + D_\rho Z) - \rho^2, \tau(X_0 + D_\rho Z))$, which is equivalent to $Z \in Q_1^{m,n,1}(0) \cap \mathcal{M}'$ and $r \in (\tau(Z) - 1, \tau(Z))$. Theorem 3.6.1 then gives

$$C_5 \geq \sup_{\mathcal{M}' \cap Q_1^{m,n,1}(0)} G_{2,\alpha}(\mathcal{M}', \cdot) d(\cdot, P_1^{m,n,1}(0)) = \sup_{\mathcal{M} \cap Q_\rho^{m,n,1}(X_0)} G_{2,\alpha}(\mathcal{M}, \cdot) d(\cdot, P_\rho^{m,n,1}(X_0)).$$

□

The next corollary should be compared Theorem 3.5 of [24].

Corollary 3.6.2. *Let $\mathcal{M}$ be a mean curvature flow satisfying Assumptions 1, 2, 3 and 5. Let X_0 lie in the closure $\bar{\mathcal{M}}$ such that $\tau(X_0) = \sup I$.*[23] *Suppose that $\rho_0 > \rho > 0$ are such that $Q_{\rho_0}^{m,n,1}(Y)$ is contained in $\Omega \times \mathbb{R}^n \times I$ and*

$$\Theta(\mathcal{M}, Y, s, \rho_0) \geq 1 - \epsilon$$

for all $Y \in Q_\rho^{m,n,1}(X_0) \cap \mathcal{M}$ and all $s \in (\tau(Y) - \rho^2, \tau(Y))$. Then $\bar{\mathcal{M}}$ will be smooth in some spacetime neighbourhood of X_0.

Under the assumptions of this corollary, our flow is given by a function u, smooth on $\Omega \times I$ and continuous on the closure, so the point X_0 is $(\hat{x}_0, u(\hat{x}_0, \tau(X_0)), \tau(X_0))$ for some $\hat{x}_0 \in \Omega$. The corollary then says that, even if $\tau(X_0) = \sup I \notin I$, there exists some $R > 0$ such that all of the derivatives of u have continuous extension to $U_R^{m,1}(\hat{x}_0, \tau(X_0)) = B_R^m(\hat{x}_0) \times (\tau(X_0) - R^2, \tau(X_0)]$.

Proof. We take a sequence $X_J \to X_0$ (as $J \to \infty$) in $\mathcal{M}$ with $\tau(X_J) < \tau(X_0)$ and with $\hat{x}_J = \hat{x}_0$. For large J, $||X_J - X_0|| < \rho/2$ and we define

$$\mathcal{M}_J = \{Y \in \mathcal{M} \mid \tau(Y) \leq \tau(X_J)\}.$$

[23] By these assumptions, $\bar{\mathcal{M}}$ is continuous at time $\tau(X_0)$ but not necessarily smooth, since we only assume X_0 to be on the closure and not necessarily on the flow itself.

Now $\Theta(\mathcal{M}_J, Y, s, \rho_0) \geq 1 - \epsilon$ for $Y \in Q^{m,n,1}_{\rho/2}(X_J) \cap \mathcal{M}_J \subset Q^{m,n,1}_{\rho}(X_0) \cap \mathcal{M}$ and $s \in (\tau(Y) - \rho^2/4, \tau(Y)) \subset (\tau(Y) - \rho^2, \tau(Y))$.[24] Then, by Corollary 3.6.1,

$$\sup_{\mathcal{M}_J \cap Q^{m,n,1}_{\rho/2}(X_J)} G_{2,\alpha}(\mathcal{M}_J, \cdot) d(\cdot, P^{m,n,1}_{\rho/2}(X_J)) \leq C_5 \tag{3.24}$$

for sufficiently large J. Now we take some sequence Y_J in $\mathcal{M}_J \cap Q^{m,n,1}_{\rho/2}(X_J)$ with $\hat{y}_J = \hat{x}_0$ and $Y_J \to X_0$. For large J we can obviously assume $d(Y_J, P^{m,n,1}_{\rho/2}(X_J)) \geq \rho/4$, and then inequality (3.24) gives $G_{2,\alpha}(\mathcal{M}_J, Y_J) \leq 4C_5/\rho$. Therefore, on the sets $U^{m,1}_{\rho/4C_5}(\hat{x}_0, \tau(Y_J))$, we have $||u||_{2,\alpha} = ||u_J||_{2,\alpha}$ uniformly bounded as $J \to \infty$. This implies a uniform bound on $||u||_{2,\alpha}$ on each of the sets $U^{m,1}_{\rho/4C_5}(\hat{x}_0, \tau(X_0)) \cap \{(\hat{x}, t) \mid t \leq \tau(Y_J)\}$, and hence on $U^{m,1}_{\rho/4C_5}(\hat{x}_0, \tau(X_0))$ since $\tau(Y_J) \to \tau(X_0)$.[25] We get smoothness from the usual parabolic differentiability theorem (Theorem B.3.3). $\square$

The following is the exact statement of the result described in the introduction. We do not assume that the domain is convex or bounded here, only that the flow is a smooth graph up to (but not including) time T with bounded gradient.

Theorem 3.6.2. *Let $\mathcal{M}$ be a mean curvature flow satisfying Assumptions 1 and 2 with $I = (0, T)$ and $|||Du|||^2 \leq 1 - \kappa$ for some $\kappa \in (0, 1)$. Then $\mathcal{M}$ can be extended smoothly to time T.*

Proof. We can extend the corresponding function u continuously to T by Proposition 3.3.3 and let $X_0 = (\hat{x}_0, u(\hat{x}_0, T), T)$ for any $\hat{x}_0 \in \Omega$. We can take a convex, bounded neighbourhood $\Omega_0 \subset \Omega$ of $\hat{x}_0$ and some $t_0 \in (0, T)$. Then the flow $\mathcal{M}_0$ given by the restriction of u to $\Omega_0 \times (t_0, T)$ will satisfy Assumptions 3 and 5. Choosing $\rho_0 > 0$ to be sufficiently small, we first apply Theorem 3.7.1 and Proposition 3.5.3 to get $\lim_{t \to T} \Theta(\mathcal{M}_0, X_0, t, \rho_0) > 1 - \epsilon$. Then we can apply Corollary 3.5.1, which allows us to use Corollary 3.6.2 to get smoothness of $\bar{\mathcal{M}}_0$ in a neighbourhood of X_0. We can do this at any $\hat{x}_0 \in \Omega$, and therefore $\mathcal{M}$ can be extended smoothly to T. $\square$

[24] We have used $Y \in Q^{m,n,1}_{\rho/2}(X_J) \Rightarrow ||X_J - Y|| < \rho/2 \Rightarrow ||Y - X_0|| \leq ||Y - X_J|| + ||X_J - X_0|| < \rho$.

[25] We have used the facts that Hölder continuity of a function on a domain implies uniform continuity, and that uniform continuity implies that the function has a unique continuous extension to the closure of its domain.

3.7 Long Proofs

Here we prove some facts that were used in this chapter. These proofs were left until now because they are fairly long and not particularly interesting, so it is likely that they would have been a distraction from more important points.

3.7.1 Lower Bound on the Limit of Θ

To apply our main regularity theorem, we need a lower bound on $\lim_{t\to\tau(X_0)}\Theta(\mathcal{M},X_0,t)$ by $1-\epsilon$. As a quick example, suppose that we have a gradient bound $|||Du|||^2 \le 1-\kappa$ strong enough that $\kappa^{m/2} > 1-\epsilon$, then we have

$$\begin{aligned}
\lim_{t\to\tau(X_0)}\Theta(\mathcal{M},X_0,t,\rho) &= \lim_{t\to\tau(X_0)}\Theta\left(D_{1/\sqrt{\tau(X_0)-t}}(\mathcal{M}-X_0),0,-1,\frac{\rho}{\sqrt{\tau(X_0)-t}}\right)\\
&\ge \lim_{t\to\tau(X_0)}\int_{B^m_{\rho/2\sqrt{\tau(X_0)-t}}(0)}\frac{1}{(4\pi)^{m/2}}e^{-|\hat{x}|^2/4}\kappa^{m/2}d\hat{x}\\
&= \kappa^{m/2}\int_{\mathbb{R}^m}\frac{1}{(4\pi)^{m/2}}e^{-|\hat{x}|^2/4}d\hat{x}\\
&= \kappa^{m/2}\\
&> 1-\epsilon,
\end{aligned}$$

where we have used Proposition 3.5.1, the fact that $\phi = 1$ in $B^m_{1/2}(0)$, the inequality $\det\hat{g}\ge\kappa^m$, and finally the usual formula for Gaussian integrals. But the assumption on the gradient in this example is not very nice and involves an unknown constant. We can do much better than this, as we will see in the next theorem.

Theorem 3.7.1. *Let $\mathcal{M}$ satisfy Assumptions 1, 2, 3 and 5, with $I=(0,T)$. For any $X_0=(\hat{x}_0,u(\hat{x}_0,T),T)\in\bar{\mathcal{M}}$ with $\hat{x}_0\in\Omega$, the limit $\lim_{t\to T}\Theta(\mathcal{M},X_0,t)$ is greater than or equal to 1.*

It is important to remember that we are not assuming the flow to be smooth on $\Omega\times(0,T]$, only continuous. The proof of this theorem is roughly the same as the proofs of similar results in [22].

Proof. We will first define a function on the flow,

$$\zeta = 1+\log(1/\kappa^{m/2})-\log(\cosh\theta),$$

where θ is the *hyperbolic angle* defined on page 3 of [17].[26] An evolution equation in [17] (see inequality (A.3) in the appendix here) tells us that

$$\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)\zeta \geq \kappa |B|^2,$$

where $|B|^2$ is the norm of the second fundamental form on the spatial slices. We note that there exist constants $C_7, C_8 > 0$ (depending on κ) such that $C_7|B|^2 \leq |D^2u|^2 \leq C_8|B|^2$.[27] Another useful fact is that, by the assumption $|||Du|||^2 \leq 1-\kappa$, there exists a constant $C_9 > 0$ (again only depending on κ) such that if $v \in \mathbb{R}^{m+n}_n$ is any tangent vector to $\mathcal{M}(t)$ then $\langle v, v\rangle \leq |v|^2 \leq C_9 \langle v, v\rangle$.[28]

We use ϕ_{ρ,X_0} from Definition 3.5.1 for small enough ρ. Equation (3.7) gives

$$\frac{d}{dt}\int_{\mathcal{M}(t)} \Phi_{X_0}\zeta\phi_{\rho,X_0} \geq \int_{\mathcal{M}(t)} \Phi_{X_0}\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)(\zeta\phi_{\rho,X_0}).$$

It is easy to check (compare to Lemma 3.14 of [5]) that we have the product rule,[29]

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)(\phi_{\rho,X_0}\zeta) &= \partial_t(\phi_{\rho,X_0}\zeta) - \mathrm{div}_{\mathcal{M}(t)}\bar{g}D(\phi_{\rho,X_0}\zeta) \\
&= \zeta\partial_t\phi_{\rho,X_0} + \phi_{\rho,X_0}\partial_t\zeta \\
&\quad -\mathrm{div}_{\mathcal{M}(t)}(\zeta\bar{g}D\phi_{\rho,X_0}) - \mathrm{div}_{\mathcal{M}(t)}(\phi_{\rho,X_0}\bar{g}D\zeta) \\
&= \zeta\left(\partial_t\phi_{\rho,X_0} - \mathrm{div}_{\mathcal{M}(t)}\bar{g}D(\phi_{\rho,X_0})\right) \\
&\quad +\phi_{\rho,X_0}\left(\partial_t\zeta - \mathrm{div}_{\mathcal{M}(t)}\bar{g}D(\zeta)\right) \\
&\quad -2\left\langle \mathrm{grad}_{\mathcal{M}(t)}\phi_{\rho,X_0}, \mathrm{grad}_{\mathcal{M}(t)}\zeta\right\rangle \\
&= \zeta\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)\phi_{\rho,X_0} + \phi_{\rho,X_0}\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)\zeta \\
&\quad -2\left\langle \mathrm{grad}_{\mathcal{M}(t)}\phi_{\rho,X_0}, \mathrm{grad}_{\mathcal{M}(t)}\zeta\right\rangle,
\end{aligned}$$

[26] At any point on the flow, $\cosh\theta$ is equal to the value of $1/\sqrt{\det\hat{g}}$ at the corresponding point in $\Omega \times I$. Therefore we have the obvious bounds on $\cosh\theta$ which follow from the bounds on Du. In particular, ζ is bounded and positive.

[27] We can write $|B|^2 = |\langle B_{ij}, B_{kl}\rangle \hat{g}^{ik}\hat{g}^{jl}|$, see [17] for details. $|D^2u|$ just denotes the Euclidean norm of D^2u, and to prove the inequality we need the fact that the eigenvalues of Du^TDu are bounded above and below thanks to the gradient bound. Compare to page 31 of [13].

[28] $\langle v, v\rangle \leq |v|^2$ is obvious. Let $v = v^i(e_i, \partial_i u)$ and then, since $|||Du|||^2 \leq 1-\kappa$, $\langle v, v\rangle = \hat{g}_{ij}v^iv^j \geq \kappa\sum_i (v^i)^2$ and $|v|^2 = v^T(I + Du^TDu)v \leq (2-\kappa)\sum_i(v^i)^2$.

[29] This is proved easily using equations (3.1) and (3.2), with the facts that $(\bar{g}Df)^\top = \mathrm{grad}_{\mathcal{M}(t)}f$ and $\mathrm{div}_{\mathcal{M}(t)}(fV) = f\mathrm{div}_{\mathcal{M}(t)}V + \left\langle V, \mathrm{grad}_{\mathcal{M}(t)}f\right\rangle$.

By Young's inequality,[30]

$$\begin{aligned}
\left\langle \mathrm{grad}_{\mathcal{M}(t)}\phi_{\rho,X_0}, \mathrm{grad}_{\mathcal{M}(t)}\zeta \right\rangle &= \left\langle \frac{\mathrm{grad}_{\mathcal{M}(t)}\phi_{\rho,X_0}}{\sqrt{\phi_{\rho,X_0}}}, \sqrt{\phi_{\rho,X_0}}\mathrm{grad}_{\mathcal{M}(t)}\zeta \right\rangle \\
&\leq \frac{1}{2\epsilon}\frac{|\mathrm{grad}_{\mathcal{M}(t)}\phi_{\rho,X_0}|^2}{\phi_{\rho,X_0}} + \frac{\epsilon}{2}\phi_{\rho,X_0}|\mathrm{grad}_{\mathcal{M}(t)}\zeta|^2 \\
&\leq \frac{1}{\epsilon}C_{10}\frac{|D\phi_{\rho,X_0}|^2}{\phi_{\rho,X_0}} + \frac{\epsilon}{2}C_9\phi_{\rho,X_0}\left\langle \mathrm{grad}_{\mathcal{M}(t)}\zeta, \mathrm{grad}_{\mathcal{M}(t)}\zeta \right\rangle,
\end{aligned}$$

where $C_{10}(\kappa) > 0$ and ϵ is any positive number. Since ϕ_{ρ,X_0} is compactly supported on the flow, Example 3.16 of [5][31] implies that $|D\phi_{\rho,X_0}|^2/\phi_{\rho,X_0} \leq 2\max|D^2\phi_{\rho,X_0}|$, where we remember that $|D^2\phi_{\rho,X_0}| < C_1/\rho^2$. By inequality (A.4), we see that $\left\langle \mathrm{grad}_{\mathcal{M}(t)}\zeta, \mathrm{grad}_{\mathcal{M}(t)}\zeta \right\rangle \leq C_{11}|B|^2$ for some constant $C_{11}(\kappa)$.[32] So there exist constants $C_{12}, C_{13}, C_{14} > 0$ (depending on κ, ρ) such that

$$\begin{aligned}
2\left\langle \mathrm{grad}_{\mathcal{M}(t)}\phi_{\rho,X_0}, \mathrm{grad}_{\mathcal{M}(t)}\zeta \right\rangle &\leq \frac{C_{12}}{\epsilon} + \epsilon C_{13}\phi_{\rho,X_0}|B|^2, \\
\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)\phi_{\rho,X_0} &\leq C_{14},
\end{aligned}$$

where we prove the second inequality as in Theorem 3.4.1. Combining all of the inequalities above,

$$\frac{d}{dt}\int_{\mathcal{M}(t)} \Phi_{X_0}\zeta\phi_{\rho,X_0} \geq \int_{\mathcal{M}(t)} \Phi_{X_0}\left(\kappa\phi_{\rho,X_0}|B|^2 - C_{15}C_{14} - \frac{C_{12}}{\epsilon} - \epsilon C_{13}\phi_{\rho,X_0}|B|^2\right),$$

where we use the fact that ζ is clearly less than or equal to some constant $C_{15}(\kappa)$. Choosing $\epsilon = \kappa/2C_{13}$ and $C_{16} = C_{15}C_{14} + C_{12}/\epsilon$, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{M}(t)} \Phi_{X_0}\zeta\phi_{\rho,X_0} &\geq \frac{\kappa}{2}\int_{\mathcal{M}(t)} \Phi_{X_0}\phi_{\rho,X_0}|B|^2 - C_{16}\int_{\mathcal{M}(t)} \Phi_{X_0} \\
&= \frac{\kappa}{2}\int_{\mathcal{M}(t)} \Phi_{X_0}\phi_{\rho,X_0}|B|^2 - C_{16}\Theta(\mathcal{M}, X_0, t).
\end{aligned}$$

We can now use this to prove the theorem. We assume that $\lim_{t\to T}\Theta(\mathcal{M}, X_0, t) < 1$ and hope to get a contradiction. So for t close enough to $\tau(X_0) = T$ (say $t \in (T-\delta, T)$ for some $\delta > 0$) we can assume that

$$\frac{d}{dt}\int_{\mathcal{M}(t)} \Phi_{X_0}\zeta\phi_{\rho,X_0} \geq \frac{\kappa}{2}\int_{\mathcal{M}(t)} \Phi_{X_0}\phi_{\rho,X_0}|B|^2 - C_{16}. \tag{3.25}$$

[30]This implies that $\langle v, w\rangle = v^i w^i - v^\gamma w^\gamma \leq |v^i||w^i| + |v^\gamma||w^\gamma| \leq \sum_i[\epsilon(v^i)^2/2 + (w^i)^2/2\epsilon] + \sum_\gamma[\epsilon(v^\gamma)^2/2 + (w^\gamma)^2/2\epsilon] = \epsilon|v|^2/2 + |w|^2/2\epsilon$.

[31]$|D\phi|^2/\phi \leq 2\max|D^2\phi|$ for compactly supported C^2 functions.

[32]We could even prove this directly, using the Schwarz inequality and $|D^2u|^2 \leq C_8|B|^2$.

We can see how this inequality is affected by parabolic dilations, D_λ for $\lambda > 1$, by noting that ζ involves first derivatives and $|B|$ involves second derivatives. To save space here, it is convenient to define

$$\begin{aligned} f(\mathcal{M}, X_0, t, \rho) &= \int_{\mathcal{M}(t)} \Phi_{X_0} \zeta \phi_{\rho, X_0}, \\ k(\mathcal{M}, X_0, t, \rho) &= \int_{\mathcal{M}(t)} \Phi_{X_0} \phi_{\rho, X_0} |B|^2, \end{aligned}$$

and the inequality above becomes

$$\frac{d}{dt} f(\mathcal{M}, X_0, t, \rho) \geq \frac{\kappa}{2} k(\mathcal{M}, X_0, t, \rho) - C_{16}. \tag{3.26}$$

But, applying the transformation formula for integrals in the usual way, we have

$$\begin{aligned} f(D_\lambda(\mathcal{M} - X_0), 0, s, \lambda\rho) &= f(D_\lambda \mathcal{M}, D_\lambda X_0, s + \tau(D_\lambda X_0), \lambda\rho) \\ &= f(\mathcal{M}, X_0, s/\lambda^2 + \tau(X_0), \rho) \\ \Rightarrow \frac{d}{ds} f(D_\lambda(\mathcal{M} - X_0), 0, s, \lambda\rho) &= \frac{1}{\lambda^2} \frac{df(\mathcal{M}, X_0, t, \rho)}{dt}\Big|_{t = s/\lambda^2 + \tau(X_0)} \\ &\geq \frac{1}{\lambda^2} \left(\frac{\kappa}{2} k(\mathcal{M}, X_0, s/\lambda^2 + \tau(X_0), \rho) - C_{16} \right) \\ &= \frac{-C_{16}}{\lambda^2} + \frac{\kappa}{2} \int_{\mathcal{M}(s/\lambda^2 + \tau(X_0))} \Phi_{X_0} \phi_{\rho, X_0} \left(\frac{1}{\lambda^2} |B|^2 \right) \\ &= \frac{-C_{16}}{\lambda^2} + \frac{\kappa}{2} \int_{D_\lambda(\mathcal{M} - X_0)(s)} \Phi_0 \phi_{\lambda\rho, 0} |B|^2 \\ &= \frac{-C_{16}}{\lambda^2} + \frac{\kappa}{2} k(D_\lambda(\mathcal{M} - X_0)(s), 0, s, \lambda\rho). \end{aligned}$$

So we have

$$\frac{d}{ds} \int_{D_\lambda(\mathcal{M} - X_0)(s)} \Phi_0 \zeta \phi_{\lambda\rho, 0} \geq -\frac{C_{16}}{\lambda^2} + \frac{\kappa}{2} \int_{D_\lambda(\mathcal{M} - X_0)(s)} \Phi_0 |B|^2 \phi_{\lambda\rho, 0}$$

for $s \in (-\lambda^2\delta, 0)$, remembering that $\lambda > 1$. We now take $\tau < \delta/2$ and integrate with respect to s over the interval $(-\delta/2 - \tau, -\delta/2)$ to get

$$\begin{aligned} \left[\int_{\mathcal{M}(s/\lambda^2 + \tau(X_0))} \Phi_{X_0} \zeta \phi_{\rho, X_0} \right]_{-\delta/2 - \tau}^{-\delta/2} &= \left[\int_{D_\lambda(\mathcal{M} - X_0)(s)} \Phi_0 \zeta \phi_{\lambda\rho, 0} \right]_{-\delta/2 - \tau}^{-\delta/2} \\ &\geq -\frac{C_{16}\tau}{\lambda^2} + \frac{\kappa}{2} \int_{-\delta/2 - \tau}^{-\delta/2} \int_{D_\lambda(\mathcal{M} - X_0)(s)} \Phi_0 |B|^2 \phi_{\lambda\rho, 0}. \end{aligned}$$

The left hand side and the first term on the right hand side clearly have limit zero as $\lambda \to \infty$,[33] therefore we must have

$$\int_{-\delta/2-\tau}^{-\delta/2} \int_{D_\lambda(\mathcal{M}-X_0)(s)} \Phi_0 |B|^2 \phi_{\lambda\rho,0} \to 0. \tag{3.27}$$

As on page 26 of [22], we can use the integral mean value theorem[34] to choose sequences $\lambda_J \to \infty$, $\tau_J \to 0$ and $s_J \in [-\delta/2 - \tau_J, -\delta/2]$ such that

$$\int_{D_{\lambda_J}(\mathcal{M}-X_0)(s_J)} \Phi_0 |B|^2 \phi_{\lambda_J\rho,0} \to 0 \text{ as } J \to \infty.$$

To do this, we first choose a sequence $\tau_J < \delta/2$ which converges to 0 and then let C_J be such that $C_J/\tau_J \to 0$. Since $\tau_J < \delta/2$, the limit (3.27) tells us that

$$\int_{-\delta/2-\tau_J}^{-\delta/2} \int_{D_\lambda(\mathcal{M}-X_0)(s)} \Phi_0 |B|^2 \phi_{\lambda\rho,0} \to 0 \quad \text{as} \quad \lambda \to \infty,$$

and therefore is less than C_J for large enough λ. For each J we choose such large λ and denote them by λ_J (we also choose these such that $\lambda_J \to \infty$). Then

$$\int_{-\delta/2-\tau_J}^{-\delta/2} \int_{D_{\lambda_J}(\mathcal{M}-X_0)(s)} \Phi_0 |B|^2 \phi_{\lambda_J\rho,0} < C_J,$$

but by the mean value theorem we have

$$\int_{-\delta/2-\tau_J}^{-\delta/2} \int_{D_{\lambda_J}(\mathcal{M}-X_0)(s)} \Phi_0 |B|^2 \phi_{\lambda_J\rho,0} = (-\delta/2 + \delta/2 + \tau_J) \int_{D_{\lambda_J}(\mathcal{M}-X_0)(s_J)} \Phi_0 |B|^2 \phi_{\lambda_J\rho,0}$$

for some $s_J \in [-\delta/2 - \tau_J, -\delta/2]$. Therefore

$$\int_{D_{\lambda_J}(\mathcal{M}-X_0)(s_J)} \Phi_0 |B|^2 \phi_{\lambda_J\rho,0} < C_J/\tau_J \to 0, \tag{3.28}$$

as expected.

We have $\delta/2 \leq |s_J| \leq \delta$, so

$$\Phi_0(\hat{x}, \tilde{x}, s_J) = \frac{\exp\left((-|\hat{x}|^2 + |\tilde{x}|^2)/4|s_J|\right)}{(4\pi|s_J|)^{m/2}} \geq \frac{\exp\left(-|\hat{x}|^2/2\delta\right)}{(4\pi\delta)^{m/2}}.$$

[33] The limit on the left hand side involves $\lim_{t\to T} f$, which must exist since $f \leq C_{15}\Theta \leq C_{15}$ for t close to T and $df/dt \geq (\kappa/2)k - C_{16}$ (which implies that $f + tC_{16}$ is monotone).

[34] $\int_a^b f = f(x)(b-a)$ for some $x \in [a,b]$.

The function $\phi_{\lambda_J\rho,0}$ is zero outside $B^m_{\lambda_J\rho}(0)\times\mathbb{R}^n$ and is equal to 1 inside $B^m_{\lambda_J\rho/2}(0)\times\mathbb{R}^n$. For any $R>0$ we can take J large enough such that $B^m_R(0)\times\mathbb{R}^n\subset B^m_{\lambda_J\rho/2}(0)\times\mathbb{R}^n$, and then we have (as $J\to\infty$)

$$\begin{aligned}\frac{\exp(-R^2/2\delta)}{(4\pi\delta)^{m/2}}\int_{D_{\lambda_J}(\mathcal{M}-X_0)(s_J)\cap B^m_R(0)\times\mathbb{R}^n}|B|^2 &\leq \int_{D_{\lambda_J}(\mathcal{M}-X_0)(s_J)}\Phi_0\phi_{\rho\lambda_J,0}|B|^2\\ &\to 0.\end{aligned}$$

Now consider the functions $\tilde{u}_J(\hat{x})$ whose graphs give the spatial slices $D_{\lambda_J}(\mathcal{M}-X_0)(s_J)$. The fact that $\lambda_J\to\infty$ tells us that, for any $R>0$, we can take J large enough such that $B^m_R(0)$ is contained in the domain of $\tilde{u}_J$. Since we also have a uniform bound on the gradients $D\tilde{u}_J$, the usual Arzela-Ascoli theorem argument gives a subsequence (which we continue to denote by $\tilde{u}_J$) converging pointwise on $\mathbb{R}^m$, and uniformly on each $B^m_R(0)$, to some limit $\tilde{u}$. As explained earlier, we have $C_7|B|^2\leq|D^2\tilde{u}_J|^2\leq C_8|B|^2$, so the inequality above gives

$$\int_{B^m_R(0)}|D^2\tilde{u}_J|^2\to 0 \text{ as } J\to\infty.$$

If we define $v^{k\gamma}_J=\partial_k\tilde{u}^\gamma_J$ and $c^{k\gamma}_J=\int_{B^m_R(0)}v^{k\gamma}_J/\mathrm{vol}(B^m_R(0))$, then the limit above tells us that

$$\int_{B^m_R(0)}|Dv^{k\gamma}_J|^2=\int_{B^m_R(0)}\sum_i(\partial_{ik}\tilde{u}^\gamma_J)^2\leq\int_{B^m_R(0)}|D^2\tilde{u}_J|^2\to 0$$

as $J\to\infty$. We can take a convergent subsequence $c^{k\gamma}_J\to c^{k\gamma}$ (since the sequence is clearly bounded due to the gradient bound on $\tilde{u}_J$) and apply the Poincaré inequality[35] to get

$$\int_{B^m_R(0)}|v^{k\gamma}_J-c^{k\gamma}_J|^2\leq C_{17}\int_{B^m_R(0)}|Dv^{k\gamma}_J|^2\to 0.$$

So $v^{k\gamma}_J-c^{k\gamma}_J\to 0$ with respect to the L^2 norm on $B^m_R(0)$. Now we can assume[36] that the derivatives of our sequence converge pointwise almost everywhere to constants.

[35] If Ω is a bounded open subset of $\mathbb{R}^m$ with Lipschitz boundary, then there exists a constant $C(\Omega,p)$ such that every function in the Sobolev space $W^{1,p}(\Omega)$ satisfies $||f-f_\Omega||_{L^p(\Omega)}\leq C(\Omega,p)||Df||_{L^p(\Omega)}$, where $u_\Omega=\int_\Omega f/\mathrm{vol}\Omega$. Here $W^{1,p}$ contains weakly differentiable functions with finite L^p norm whose weak derivatives also have finite L^p norm.

[36] See Theorem 19.12 of [9], which says that L^p convergence implies pointwise convergence almost everywhere.

These constants will be the weak derivatives of $\tilde{u}$,[37] which therefore must be linear. Remember that we are assuming $\lim_{t\to T}\Theta(\mathcal{M}, X_0, t) < 1$, so we can write

$$\begin{aligned}\lim_{J\to\infty}\Theta(D_{\lambda_J}(\mathcal{M}-X_0),0,s_J) &= \lim_{J\to\infty}\Theta(\mathcal{M},X_0,T+s_J/\lambda_J^2)\\ &= \lim_{t\to T}\Theta(\mathcal{M},X_0,t)\\ &= 1-\eta \end{aligned}\tag{3.29}$$

for some $\eta > 0$. Let $\tilde{M}_{J,R}$ be the graph of $\tilde{u}_J|_{B_R^m(0)}$ and let $\tilde{M}_R$ be the graph of $\tilde{u}|_{B_R^m(0)}$. Since we know that Θ is equal to 1 on non-moving planes, and since the graph of $\tilde{u}$ will be a spatial slice of a non-moving plane, we can choose $\int_{\tilde{M}_R}\Phi_0(\cdot,-\delta/2)$ to be as close to 1 as we like by taking R large. We choose R such that $1-\eta/2 < \int_{\tilde{M}_R}\Phi_0(\cdot,-\delta/2)$. We also have

$$\begin{aligned}\Theta(D_{\lambda_J}(\mathcal{M}-X_0),0,s_J) &> \int_{\tilde{M}_{J,R}}\Phi_0(\cdot,s_J)\\ &= \int_{B_R^m(0)}\frac{\exp\left(\frac{-|\hat{x}-0|^2+|\tilde{u}_J(\hat{x})-0|^2}{4(0-s_J)}\right)}{(4\pi(0-s_J))^{m/2}}\sqrt{\det\hat{g}(D\tilde{u}_J)}d\hat{x},\end{aligned}$$

where the terms in the integrand all converge pointwise on the set $B_R^m(0)$, with $\tilde{u}_J$ converging uniformly. Then, by the dominated convergence theorem and since $s_J\to-\delta/2$,

$$\Theta(D_{\lambda_J}(\mathcal{M}-X_0),0,s_J) > \int_{\tilde{M}_{J,R}}\Phi_0(\cdot,s_J)\to\int_{\tilde{M}_R}\Phi_0(\cdot,-\delta/2) > 1-\eta/2.$$

This gives a contradiction by equation (3.29). □

3.7.2 Properties of $G_{2,\alpha}$

Although the properties of $G_{2,\alpha}$ can easily be understood by comparison with White's $K_{2,\alpha}$ norm, we will explain in more detail here just so that everything is clear. First we need to understand exactly what it means for $G_{2,\alpha}$ to be bounded from above or below. Let $G_{2,\alpha}(\mathcal{M},Y) < \Lambda$ for some fixed $Y = (\hat{y},\tilde{y},s)\in\mathcal{M}$ and some positive constant Λ. By the definition of $G_{2,\alpha}$, this implies that

$$[Du_{\Lambda,Y}]_\alpha + [D^2u_{\Lambda,Y}]_\alpha + \sup|D^2u_{\Lambda,Y}| + [\partial_t u_{\Lambda,Y}]_\alpha + \sup|\partial_t u_{\Lambda,Y}| \le 1$$

[37]See Theorem 20.7 in [9], which says that if a sequence of smooth functions f_J converges in $L^p(\Omega)$ to some f, and if $\partial_i f_J$ converges in $L^p(\Omega)$ to some v, then v is the weak derivative of f.

on the set $U^{m,1} \cap D_\Lambda(\Omega \times I - (\hat{y}, s))$, where $u : \Omega \times I \to \mathbb{R}^n$ is such that $\mathcal{M}$ is the graph of u and $u_{\Lambda,Y}(\cdot,\cdot) = \Lambda(u(\cdot/\Lambda + \hat{y}, \cdot/\Lambda^2 + s) - \tilde{y})$. But, for example, we have $\partial_t u_{\Lambda,Y}(\cdot,\cdot) = (1/\Lambda)\partial_t u(\cdot/\Lambda + \hat{y}, \cdot/\Lambda^2 + s)$ and therefore $\sup_{U^{m,1}} |\partial_t u_{\Lambda,Y}| = (1/\Lambda)\sup_{B^m_{1/\Lambda}(\hat{y})\times(s-1/\Lambda^2,s]} |\partial_t u|$. Similar reasoning applies to each of the other terms in the inequality above, giving

$$\Lambda^{-\alpha}[Du]_\alpha + \Lambda^{-1-\alpha}[D^2u]_\alpha + \Lambda^{-1}\sup|D^2u| + \Lambda^{-1-\alpha}[\partial_t u]_\alpha + \Lambda^{-1}\sup|\partial_t u| \leq 1 \quad (3.30)$$

on the set $B^m_{1/\Lambda}(\hat{y}) \times (s - 1/\Lambda^2, s] \cap \Omega \times I$. In particular, we will have

$$[Du]_\alpha + [D^2u]_\alpha + \sup|D^2u| + [\partial_t u]_\alpha + \sup|\partial_t u| \leq C(\Lambda) \quad (3.31)$$

on $B^m_{1/\Lambda}(\hat{y}) \times (s - 1/\Lambda^2, s] \cap \Omega \times I$, for some constant $C(\Lambda)$ dependent only on Λ.

Now suppose that $G_{2,\alpha}(\mathcal{M}, Y) \leq \Lambda$ for every Y in a compact subset K of space-time, then inequality (3.31) gives a bound $[Du]_\alpha + [D^2u]_\alpha + \sup|D^2u| + [\partial_t u]_\alpha + \sup|\partial_t u| \leq C(\Lambda, K)$ on the projection of K onto $\mathbb{R}^{m,1}$, where $C(\Lambda, K)$ is a constant depending only on Λ and K. Assume further that the origin lies in both $\mathcal{M}$ and K, then inequality (3.12) and the spacelike assumption (that $|||Du||| < 1$) tell us that

$$|u(\hat{x}, t) - 0| = |u(\hat{x}, t) - u(0, 0)| \leq 1.|\hat{x} - 0| + |t - 0|.C(\Lambda, K) \leq C'(\Lambda, K)$$

for some constant $C'(\Lambda, K)$ depending only on Λ and K. Combining all of these inequalities gives a bound on $||u||_{2,\alpha}$ (on the projection of K onto $\mathbb{R}^{m,1}$) which is dependent only on K and Λ.

By similar reasoning, a lower bound $G_{2,\alpha}(\mathcal{M}, Y) > \Lambda$ at a point Y corresponds to a lower bound on the $C^{2,\alpha}$ norm of u. In particular we have

$$[Du]_\alpha + [D^2u]_\alpha + \sup|D^2u| + [\partial_t u]_\alpha + \sup|\partial_t u| \geq C''(\Lambda) \quad (3.32)$$

on the set $B^m_{1/\Lambda}(\hat{y}) \times (s - 1/\Lambda^2, s] \cap \Omega \times I$, where $C''(\Lambda) > 0$ is a constant depending only on Λ.

We only need the following fact in the proof of Theorem 3.6.1. Here we will go through the details of the Arzela-Ascoli theorem for sequences of flows with $G_{2,\alpha}$ bounded on compact sets.

Proposition 3.7.1. *The sequence $\tilde{\mathcal{M}}_J \cap \{X \mid \tau(X) \leq 0\}$, from the proof of Theorem 3.6.1, has a convergent subsequence (this is parabolic C^2 convergence on compact subsets). The limit is a smooth flow $\mathcal{M}'$ satisfying Assumptions 2, 3 and 4 with $T = 0$. Also, $\mathcal{M}'$ has uniformly bounded mean curvature vector.*

Proof. Let $\tilde{u}_J$, $\tilde{\mathcal{M}}_J$, etc. be exactly as in the proof of Theorem 3.6.1. Then, since $\lambda_J \to \infty$ and $\sup \tilde{I}_J \geq 0$, any compact subset of $\mathbb{R}^m \times (-\infty, 0]$ will be contained in the domain of $\tilde{u}_J$ for large enough J. By inequality (3.22), $G_{2,\alpha}(\tilde{\mathcal{M}}_J, \cdot)$ will be uniformly bounded on compact subsets of spacetime with $\tau(X) \leq 0$ as $J \to \infty$. Therefore we get uniform bounds on $||\tilde{u}_J||_{2,\alpha}$ on compact subsets of $\mathbb{R}^m \times (-\infty, 0]$ (i.e. bounds which are independent of J, for all large enough J, but do depend on the subset itself and the $G_{2,\alpha}$ bound). We can use this to prove convergence of a subsequence by following the same steps as in the proof of the Arzela-Ascoli theorem.

We use the Cantor diagonalization process to choose a convergent subsequence (compare to the proof of Theorem 5.20 of [9]). Take a countable dense sequence $\{(\hat{x}_J, t_J)\}_{J \in \mathbb{N}}$ of $\mathbb{R}^m \times (-\infty, 0]$ (we can do this since Euclidean spaces are second countable). Take the point $(\hat{x}_1, t_1)$ and a subset $D_1 \subset \mathbb{R}^m \times (-\infty, 0]$ containing this point and the origin. By the reasoning above, we can choose J large enough such that $||\tilde{u}_J||_{2,\alpha}$ is bounded independently of J on D_1. In particular, the sequence $\tilde{u}_J$ (and it derivatives up to second order) will be uniformly bounded on D_1. Since the sequence is bounded (independent of J) we can take a subsequence $\tilde{u}_{1,J}$ for which $\tilde{u}_{1,J}(\hat{x}_1, t_1)$ converges. Iteratively, we choose a subsequence $\tilde{u}_{K,J}$ of $\tilde{u}_{K-1,J}$ such that $\tilde{u}_{K,J}(\hat{x}_K, t_K)$ converges. $\tilde{u}_{K,J}$ is a subsequence of $\tilde{u}_{K-1,J}$, so inductively the sequences $\tilde{u}_{K,J}(\hat{x}_1, t_1), \ldots, \tilde{u}_{K,J}(\hat{x}_{K-1}, t_{K-1})$ converge as $J \to \infty$. Then $\tilde{u}_{K,J}(\hat{x}_H, t_H)$ converges as $J \to \infty$ for all $H \leq K$, so the diagonal subsequence $\tilde{u}_{J,J}(\hat{x}_H, t_H)$ converges as $J \to \infty$ for every H.

Now we claim that, on any compact subset D of $\mathbb{R}^m \times (-\infty, 0]$, the sequence $\tilde{u}_{K,K}$ converges uniformly on D as $K \to \infty$. Suppose that we are given $\epsilon > 0$ and $(\hat{z}, r) \in D$. By the $C^{2,\alpha}$ bound, we already know that the original sequence is uniformly equicontinuous on D (for large K) and therefore so is the subsequence

$\tilde{u}_{K,K}$. We use this to choose $\delta > 0$ such that $||(\hat{x},t) - (\hat{y},s)|| < \delta$ in D implies that $|\tilde{u}_{K,K}(\hat{x},t) - \tilde{u}_{K,K}(\hat{y},s)| < \epsilon/3$. Since $\{(\hat{x}_J,t_J)\}$ is dense in $\mathbb{R}^m \times (-\infty,0]$, we can choose some $(\hat{x}_J,t_J)$ such that $||(\hat{z},r) - (\hat{x}_J,t_J)|| < \delta$. Then we have $|\tilde{u}_{K,K}(\hat{z},r) - \tilde{u}_{L,L}(\hat{z},r)| \leq |\tilde{u}_{K,K}(z,r) - \tilde{u}_{K,K}(\hat{x}_J,t_J)| + |\tilde{u}_{K,K}(\hat{x}_J,t_J) - \tilde{u}_{L,L}(\hat{x}_J,t_J)| + |\tilde{u}_{L,L}(\hat{x}_J,t_J) - \tilde{u}_{L,L}(\hat{z},r)|$. The first and last term on the right hand side here are clearly $< \epsilon/3$, and the second term is $< \epsilon/3$ (for large enough L and K) since we know that $\tilde{u}_{L,L}(\hat{x}_J,t_J)$ converges for every J as $L \to \infty$. Hence $|\tilde{u}_{K,K}(\hat{z},r) - \tilde{u}_{L,L}(\hat{z},r)| < \epsilon$ for large K and L, so the sequence $\tilde{u}_{K,K}(\hat{z},r)$ is Cauchy and must converge. This applies at any point $(\hat{z},r)$ of D and gives us pointwise convergence of the diagonal subsequence on D. The uniform equicontinuity of the sequence and the fact that D is compact imply that the convergence is also uniform on D.

Again by the $C^{2,\alpha}$ bounds, we have uniform boundedness and equicontinuity of all sequences of derivatives up to second order (in the parabolic sense). We can therefore repeat this process, taking subsequences for which each of these derivatives converge uniformly on compact subsets. The result is a subsequence which converges in C^2 (parabolically) on compact subsets to some function u' defined on $\mathbb{R}^m \times (-\infty,0]$. By the fact that the $C^{2,\alpha}$ bounds are independent of all large enough J on compact subsets, it is clear that this limit will also be locally $C^{2,\alpha}$ and will therefore be smooth by the standard differentiability theorems for parabolic equations (since the C^2 convergence implies that the equation from Proposition 3.3.1 will hold for u'). This smoothness follows from Theorem B.3.3 by, as in the elliptic case, using induction (if u' is $C^{k,\alpha}$ for $k \geq 2$, it satisfies a strictly parabolic linear system with $C^{k-1,\alpha}$ coefficients and must be $C^{k+1,\alpha}$).

For any $X = (\hat{x},\tilde{x},t)$ with $\tau(X) \leq 0$ we use inequality (3.22) and take J large enough such that $d(0, P^{m,n,1}_{\lambda_J}(-D_{\lambda_J}Y_J)) > 2||X||$. Then we have $G_{2,\alpha}(\tilde{\mathcal{M}}_J, X) < 4$. Then inequality (3.30) implies that $|D^2\tilde{u}_J|(\hat{x},t) + |\partial_t\tilde{u}_J|(\hat{x},t) < 4$ as $J \to \infty$, and then the C^2 convergence to u' implies that $|D^2u'| + |\partial_t u'| \leq 4$. This applies at every $(\hat{x},t) \in \mathbb{R}^m \times (-\infty,0]$. Obviously we also have $|||Du'|||^2 \leq 1 - \kappa$, and therefore we must have a uniform bound on the mean curvature vector of the spatial slices of the graph of u', since this vector is given by $(0, g^{ij}(Du')\partial_{ij}u')^{\perp}$ (see the proof of Proposition A.1.1). $\square$

Chapter 4

Final Comments

Our main theorems for the maximal graph system say that a solution to the Dirichlet problem will exist for boundary data with C^2 norm less than some constant. In the case of 2 dimensions, we only need to choose the constant such that a gradient estimate will hold. For arbitrary dimension and codimension, we need to choose the constant to be even smaller, such that both the gradient estimate and the $C^{1,\alpha}$ estimate will hold. Obviously, it would be nice to improve our result in such a way that, even for arbitrary dimension and codimension, we only need to choose the constant small enough to get the gradient estimate. Since such a result (in the case of the minimal graph system) is the goal of [23], it makes sense to attempt to follow the same steps. The proof there uses two main tools. The first is a gradient estimate and the second is White's regularity theorem. We have already proved that we have both of these in the case of spacelike mean curvature flows in semi-Euclidean spaces.

In the proof of the main theorem in [23], White's theorem is used to get $C^{2,\alpha}$ estimates at time T on flows that exist on $(0,T)$. This allows a smooth extension of the flow to T, and is used to prove long time existence. However, there is a problem with this step, since White's theorem only gives *local* estimates. These estimates are enough to extend to T, but are not enough to allow the application of short time existence at T to extend further by some fixed time. This would require at least a *uniform* $C^{1,\alpha}$ estimate over all of Ω. Therefore [23] fails to prove its claim that the minimal graph Dirichlet problem is solvable when Ω is convex and $8m\,diam\Omega \sup_\Omega |||D^2\phi||| + \sqrt{2}\sup_{\partial\Omega}|||D\phi||| < 1$. After communicating with Professor Wang, he eventually agreed that this gap in his proof exists. He also suggested an alternative method to get a similar existence theorem. Our proofs of the $C^{1,\alpha}$ estimates used in Theorem 2.4.1 are based on this suggested method. These proofs could easily be repeated in minimal graph case to prove:

Theorem 4.0.2. *Let Ω be a bounded, smooth and convex domain in $\mathbb{R}^m$. There is a positive constant C (depending on Ω, m, n) such that there exists a smooth solution to the Dirichlet problem, for the minimal graph system with boundary values $\phi|_{\partial\Omega}$, whenever the function $\phi : \bar{\Omega} \to \mathbb{R}^n$ is smooth with C^2 norm less than C.*

This would be proved by using the condition on ϕ to apply the gradient estimate of [23], allowing us to assume that the gradient of any solution would be small enough that a $C^{1,\alpha}$ estimate holds (as in Lemma 2.4.2), giving existence of a solution in the usual way. This assumption on ϕ is stronger than in Wang's original claim, so this theorem is weaker. To prove the original claim, we would probably need to apply the method used in [23] but with the help of some boundary regularity theorem to get the estimates needed for long time existence. Examples of such regularity theorems, using a modified Gaussian density at the boundary, can be seen in [24]. It is not clear exactly how we could apply these to this particular problem.

This leaves us with some obvious questions. Can White's boundary regularity theorems be applied to correct Wang's proof? How would we get the required estimates on the modified Gaussian density at the boundary? Would this need stronger estimates on the gradient? Do we need extra assumptions on the boundary data (e.g. a compatibility condition), weakening the result? If so, can the boundary regularity theorems be improved in such a way that we will not need these assumptions? These questions also apply to the semi-Euclidean case. In fact, it is not known if such boundary regularity theorems even exist in the semi-Euclidean case. The answers to these questions would provide applications for the local regularity theorem proved here, and would possibly lead to stronger existence theorems for both the minimal and maximal graph problems.

Appendix A

Semi-Riemannian Manifolds

A.1 Basic Definitions and Facts

When dealing with problems in semi-Euclidean spaces, it is useful to know some basic definitions and facts related to semi-Riemannian manifolds. We will give some here, and more can be seen in [21]. First, we note that a *semi-Riemannian* manifold is defined to be a pair (M, g) where M is a smooth manifold and g is a metric tensor (a symmetric nondegenerate $(0, 2)$ tensor field of constant index) defined on M. The most obvious examples are the semi-Euclidean spaces. We say that a tangent vector v to a semi-Riemannian manifold is:

- *spacelike* if $g(v, v) > 0$ or $v = 0$,
- *null* if $g(v, v) = 0$ but $v \neq 0$,
- *timelike* if $g(v, v) < 0$.

If $(N, \bar{g})$ is a semi-Riemannian manifold, and $M \subset N$ is a submanifold of N with inclusion map $f : M \to N$, then we can take the pullback $g = f^*\bar{g}$ in the usual way. If g is a metric tensor on M, then we get a *semi-Riemannian submanifold* (M, g), where g is called the induced metric on M. If we denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections (defined as for Riemannian manifolds) corresponding to the metrics g and $\bar{g}$ respectively, then we have $\nabla_V W = (\bar{\nabla}_V W)^\top$ for any pair of tangent vector fields V, W on M.[1] We can define the *second fundamental form* B on M by taking difference of the two Levi-Civita connections,

$$B(V,W) = \bar{\nabla}_V W - \nabla_V W = (\bar{\nabla}_V W)^{\perp},$$

which is bilinear and symmetric in V and W. The *mean curvature vector* of M is the normal vector field defined by taking the trace of B with respect to the induced metric g,

$$H = \mathrm{trace}_g B = g^{ij} B\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

If M is a spacelike manifold (i.e. if the induced metric is positive definite, so that every tangent vector is spacelike) and $H = 0$ at every point on M, then M is a *maximal submanifold* of N.

On a spacelike submanifold M with coordinates x^i, we can define the gradient, divergence and Laplace operator on M in the usual way (see chapter 3 of [21]), and we have the formulae

$$\begin{aligned}
\mathrm{grad}_M f &= g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, \\
\mathrm{div}_M V &= g^{ij} g\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^j}} V\right) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g} V^j\right), \\
\Delta_M f &= \mathrm{div}_M \mathrm{grad}_M f \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j}\left(\sqrt{\det g} g^{ij} \frac{\partial f}{\partial x^i}\right),
\end{aligned} \tag{A.1}$$

for any smooth function $f : M \to \mathbb{R}$ and vector field $V = V^i \partial/\partial x^i$. We will sometimes use the fact that, given a vector field W on M, the component tangential to M is given by

$$W^{\top} = \bar{g}\left(W, \partial/\partial x^i\right) g^{ij} \partial/\partial x^j.$$

With these formulae, we can prove the following useful fact. We will frequently need some of the equations that appear in its proof.

[1] For a tangent vector V to N, we denote by $V^{\top}$ the component tangential to M and by $V^{\perp}$ the component normal to M (with respect to the metric $\bar{g}$).

Proposition A.1.1. *Let Ω be a domain in $\mathbb{R}^m$ and let $F : \Omega \to \mathbb{R}^{m+n}_n$ be a smooth embedding such that $M = F(\Omega)$ is a spacelike submanifold. Then the mean curvature at each $p \in M$ is given by $H(p) = \Delta_M F(p)$, where Δ_M is the induced Laplace operator on M.*

Proof. We first prove that $\Delta_M F$ is a normal vector. We already know that

$$\begin{aligned} \Delta_M F &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial F}{\partial x^j} \right) \\ &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \right) \frac{\partial F}{\partial x^j} + g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j}, \end{aligned} \tag{A.2}$$

where the first term on the right hand side is a tangent vector. For each of the tangent vector fields $\partial F / \partial x^k$,

$$\begin{aligned} \left\langle \Delta_M F, \frac{\partial F}{\partial x^k} \right\rangle &= \frac{1}{\sqrt{\det g}} \left\langle \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial F}{\partial x^j} \right), \frac{\partial F}{\partial x^k} \right\rangle \\ &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left\langle \sqrt{\det g} g^{ij} \frac{\partial F}{\partial x^j}, \frac{\partial F}{\partial x^k} \right\rangle - g^{ij} \left\langle \frac{\partial F}{\partial x^j}, \frac{\partial^2 F}{\partial x^i \partial x^k} \right\rangle, \end{aligned}$$

and we can apply the usual formula for differentiating determinants[2] to the first term here, which will be

$$\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (\sqrt{\det g} \delta_{ik}) = \frac{1}{2 \det g} \det g g^{ij} \frac{\partial}{\partial x^k} g_{ij} = g^{ij} \left\langle \frac{\partial^2 F}{\partial x^i \partial x^k}, \frac{\partial F}{\partial x^j} \right\rangle.$$

This cancels the second term in the equation above, giving $\langle \Delta_M F, \partial F / \partial x^k \rangle = 0$ for all k, and therefore $\Delta_M F$ is normal. By definition,

$$H = g^{ij} \left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j} \right)^{\perp} = \left(g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j} \right)^{\perp}.$$

Combining this with equation (A.2), and the fact that $\Delta_M F$ is a normal vector field, gives

$$\Delta_M F = (\Delta_M F)^{\perp} = \left(g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j} \right)^{\perp} = H,$$

at each point of M. $\square$

[2] $\partial(\det g)/\partial s = (\det g) g^{ij} \partial g_{ij} / \partial s$

A.2 Spacelike Graphic Mean Curvature Flows

Since we make use of several facts from [17], it will be convenient to state them here. In [17], graphic mean curvature flows in semi-Riemannian product manifolds are considered. Given two Riemannian manifolds (Σ_1, g_1) and (Σ_2, g_2), we define the product manifold $N = \Sigma_1 \times \Sigma_2$ with semi-Riemannian metric $\bar{g} = g_1 - g_2$. For the graph $M = \{(p, u(p)) \mid p \in \Sigma_1\}$ in N, of a smooth function $u : \Sigma_1 \to \Sigma_2$, we can take the induced metric $g = g_1 - u^* g_2$ on M and define the *hyperbolic angle* θ by

$$\cosh\theta = \frac{1}{\sqrt{\det(g_1 - u^* g_2)}}.$$

Obviously, if we take $\Sigma_1 = \Omega$ and $\Sigma_2 = \mathbb{R}^n$ with the Euclidean metrics, this includes the case of graphs in the flat semi-Euclidean spaces that we are interested in, and $\cosh\theta$ corresponds to the quantity $1/\sqrt{\det(I - Du^T Du)}$. In section 4 of [17], an evolution equation for $\log\cosh\theta$ on a mean curvature flow is proved which, in the case of semi-Euclidean spaces, has the form

$$\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)\log\cosh\theta = -|B|^2 + \sum_{k,i}\lambda_i^2 (h_{ik}^{m+i})^2 + 2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{m+j} h_{jk}^{m+i},$$

where B is the second fundamental form, λ_i^2 are the eigenvalues of $Du^T Du$, and h_{ij}^{γ} are the components of B with respect to some orthonormal frame. Under the assumption that we have $\lambda_i^2 \leq 1 - \delta$ for each i, with $\delta \in (0,1)$ constant, it is shown in section 5 of [17] that this evolution equation implies

$$\left(\frac{d}{dt} - \Delta_{\mathcal{M}(t)}\right)\log\cosh\theta \leq -\delta|B|^2. \tag{A.3}$$

We also use the following two facts from sections 3 and 5 (respectively) of [17],

$$\frac{\left\langle \operatorname{grad}_{\mathcal{M}(t)}\cosh\theta, \operatorname{grad}_{\mathcal{M}(t)}\cosh\theta\right\rangle}{\cosh^2\theta} = \sum_k\left(\sum_i(\lambda_i h_{ik}^{m+i})\right)^2 \quad\text{and}\quad |B|^2 \geq \sum_{i,j,k}(h_{ik}^{m+j})^2,$$

to easily see that, when $\lambda_i^2 < 1 - \delta$,

$$\left\langle \operatorname{grad}_{\mathcal{M}(t)}\log\cosh\theta, \operatorname{grad}_{\mathcal{M}(t)}\log\cosh\theta\right\rangle \leq C|B|^2, \tag{A.4}$$

where C is some constant depending on δ.

Appendix B

Second Order Elliptic and Parabolic PDEs

B.1 Notation

On any domain Ω in a Euclidean space, or its closure $\bar{\Omega}$, we define:

- $C^k(\Omega)$, the set of functions $u : \Omega \to \mathbb{R}$ with all derivatives of order $\leq k$ continuous on Ω (for $k = 0, 1, 2, \ldots$, or $k = \infty$).

- $C^k(\bar{\Omega})$, the set of functions in $C^k(\Omega)$ whose derivatives of order $\leq k$ have continuous extension to $\bar{\Omega}$.[1]

When $u \in C^k(D)$ for some set D, we say that u is C^k on D (or smooth on D if $k = \infty$). For parabolic problems, time derivatives are considered to be second order.

[1]We will frequently use the fact that any uniformly continuous function $u : \Omega \to \mathbb{R}^n$ has a unique continuous extension to $\bar{\Omega}$ (in particular, we will apply this to uniformly Lipschitz or Hölder continuous functions).

B.2 Elliptic Equations

All of the definitions and facts in this section come from [10]. We will consider second order operators Q of the form

$$Qu(x) = a^{ij}(x, u, Du)\frac{\partial^2 u}{\partial x^i \partial x^j} + b^i(x, u)\frac{\partial u}{\partial x^i},$$

where a^{ij} is symmetric and $x = (x^1, \ldots, x^m)$ lies in a domain Ω of $\mathbb{R}^m$ for $m \geq 2$. We say that the operator is *elliptic* if the matrix a^{ij} is positive definite everywhere. We say that it is *linear* if a^{ij} and b^i are independent of u and Du, and that it is *quasilinear* otherwise. One of the most important tools when dealing with elliptic operators is the *maximum principle*:

Theorem B.2.1. *Let Q be a linear elliptic operator on a bounded domain Ω. Suppose that $Qu \geq 0$ (≤ 0) in Ω, with $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then the maximum (minimum) of u is achieved on the boundary $\partial\Omega$.*

To understand other useful facts, we need to define the (elliptic) Hölder spaces. For $\alpha \in (0, 1)$, we say that $u : \bar{\Omega} \to \mathbb{R}$ is *Hölder continuous* with exponent α in $\bar{\Omega}$ if

$$[u]_\alpha = \sup_{x \neq y \text{ in } \bar{\Omega}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty.$$

$C^{k,\alpha}(\bar{\Omega})$ is the subset of $C^k(\bar{\Omega})$ containing functions whose derivatives up to k-th order are Hölder continuous with exponent α. We take the norm[2]

$$||u||_{k,\alpha} = ||u||_{C^{k,\alpha}(\bar{\Omega})} = \sum_{j=0}^{k} \sup_{\Omega} |D^j u| + [D^k u]_\alpha.$$

on this space. With this norm, the space $C^{k,\alpha}(\bar{\Omega})$ is a Banach space (a complete normed linear space). We say that a function is *locally* $C^{k,\alpha}$ on some set if it is $C^{k,\alpha}$ on compact subsets. We say that a domain is $C^{k,\alpha}$ if, at each point of the boundary, there is some neighbourhood in which the boundary is the graph of a $C^{k,\alpha}$ function of $m-1$ coordinates. The next three theorems all come from chapter 6 of [10]. First

[2]Note that the definition of this norm varies in many of the textbooks that we will refer to (see [10], [14], etc.), but these norms are all equivalent. Therefore a bound on one version of the norm is equivalent to a bound on any other version.

we note that an operator Q is called *strictly elliptic* if $a^{ij}\zeta_i\zeta_j \geq \lambda|\zeta|^2$, for all $\zeta \in \mathbb{R}^m$, for some positive constant λ.

Theorem B.2.2. *Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^m$ and let $u \in C^{2,\alpha}(\bar{\Omega})$ be a solution to $Qu = 0$ in Ω with $u = \phi$ on $\partial\Omega$, for some linear strictly elliptic operator Q. Then $||u||_{2,\alpha} \leq C$, where C is some constant depending on $\sup_\Omega |u|$, $||\phi||_{2,\alpha}$, m, α, λ, Ω and the $C^{0,\alpha}$ norm of the coefficients of Q.*

This is called a *Schauder estimate*. The following is the standard existence theorem for linear elliptic equations.

Theorem B.2.3. *Let Q be a strictly elliptic linear operator defined on a bounded $C^{2,\alpha}$ domain Ω. Suppose that the coefficients of Q are in $C^{0,\alpha}(\bar{\Omega})$, and let $\phi \in C^{2,\alpha}(\bar{\Omega})$. Then there is a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ to $Qu = 0$ in Ω with $u = \phi$ on $\partial\Omega$.*

We will also need the following differentiability theorem for solutions.

Theorem B.2.4. *Let Ω be a bounded $C^{k+2,\alpha}$ domain and let $\phi \in C^{k+2,\alpha}(\bar{\Omega})$. Let Q be a strictly elliptic linear operator on Ω with coefficients in $C^{k,\alpha}(\bar{\Omega})$. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $Qu = 0$ in Ω and $u = \phi$ on the boundary, then $u \in C^{k+2,\alpha}(\bar{\Omega})$.*

It is worth mentioning the following existence theorem for quasilinear problems (even though we will not apply it directly) since it makes clear the usual method used to solve Dirichlet problems for single equations. This is a reduced version of Theorem 11.8 of [10], and it is proved by applying the Schauder fixed point theorem to Hölder spaces.

Theorem B.2.5. *Let Ω be a bounded $C^{2,\alpha}$ domain. Let Q be a quasilinear elliptic operator on Ω with $b^i = 0$ and $a^{ij} \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^m)$. Let $\phi \in C^{2,\alpha}(\bar{\Omega})$. Suppose that, for some constant M, the estimate $||u||_{1,\alpha} < M$ holds for any u with $Qu = 0$ in Ω and $u = \sigma\phi$ on $\partial\Omega$ for some $\sigma \in [0,1]$. Then there exists $u \in C^{2,\alpha}(\bar{\Omega})$ with $Qu = 0$ in Ω and $u = \phi$ on $\partial\Omega$.*

It is clear from this that we need a priori $C^{1,\alpha}$ estimates to solve a quasilinear Dirichlet problem. The usual method involves splitting this estimate into four parts.

First we estimate $\sup_\Omega |u|$ in terms of the boundary data using the maximum principle. Then we use conditions on the domain and boundary data to get a boundary gradient estimate of $|Du|$ on $\partial\Omega$. We then extend this to an estimate for $|Du|$ on all of Ω, before finally getting an a priori bound on the $C^{1,\alpha}$ norm in terms of the estimates that we already have. There are standard $C^{1,\alpha}$ estimates for single equations, but such estimates are much more difficult to get for systems.

B.3 Parabolic Equations

There are many similarities between elliptic and parabolic equations. For example, we get parabolic versions of the maximum principle and the Schauder estimates. We will avoid going into too much detail for quasilinear parabolic equations, but we will state a very simple existence theorem that gives an example of short time existence for a boundary value problem. We will consider second order operators Q of the form

$$Qu(x,t) = \frac{\partial u}{\partial t} - a^{ij}\left(x,t,u,Du\right)\frac{\partial^2 u}{\partial x^i \partial x^j} + b^i\left(x,t,u\right)\frac{\partial u}{\partial x^i},$$

where a^{ij} is symmetric, $x = (x^1, \ldots, x^m)$ lies in a domain Ω of $\mathbb{R}^m$ for $m \geq 2$, and t lies in an interval in $\mathbb{R}$. We say that the operator is *parabolic* if the matrix a^{ij} is positive definite everywhere. We say that it is *linear* if a^{ij} and b^i are independent of u and its derivatives, and that it is *quasilinear* otherwise. Now we have the parabolic maximum principle (see Theorem 8.1.2 of [14], for example):

Theorem B.3.1. *Let $\Omega \times (0,T)$ be bounded. Suppose that a real valued function u is continuous on the closure of $\Omega \times (0,T)$ and C^2 on the interior. If Q is a linear parabolic operator on this domain and if $Qu \leq 0$ (≥ 0), then the maximum (minimum) of u is achieved on the parabolic boundary, $\partial\Omega \times (0,T) \cup \bar{\Omega} \times \{0\}$.*

It is important to note that, for parabolic problems, we treat the time (t) derivative in the same way as we treat the second order space (x) derivatives. So when we say u is C^2, as in the theorem above, we mean C^1 with respect to t and C^2 with respect to x. On a subset U of the spacetime $\mathbb{R}^{m,1} = \mathbb{R}^m \times \mathbb{R}$, taking the parabolic distance $||(x,t)|| = \max\{|x|, |t|^{1/2}\}$, we define the parabolic Hölder norms (for non-negative integers k, and $0 < \alpha < 1$)

$$||f||_{k,\alpha} = ||f||_{C^{k,\alpha}(U)} = \sum_{j+2h\leq k} ||D^j \left(\partial/\partial t\right)^h f||_{0,\alpha},$$

where

$$[f]_\alpha = \sup_{(x,t)\neq(y,s) \text{ in } U} \frac{|f(x,t) - f(y,s)|}{||(x,t)-(y,s)||^\alpha} \quad \text{and} \quad ||f||_{0,\alpha} = \sup_{(x,t)\in U} |f(x,t)| + [f]_\alpha.$$

The parabolic Hölder space $C^{k,\alpha}(U)$ of functions with finite $||f||_{k,\alpha}$ norm on $\bar{U}$ will be a Banach space with this norm. We say that a function is locally $C^{k,\alpha}$ on U if this norm is finite on compact subsets of U (but this notation will not cause confusion because it will always be clear whether we are dealing with elliptic or parabolic problems). As in the elliptic case, we have Schauder estimates. We only use a local version of these estimates for constant coefficient parabolic equations, which will follow directly from Theorem 8.11.1 of [14].

Theorem B.3.2. *Let $Q = \partial/\partial t - a^{ij}\partial^2/\partial x^i \partial x^j$ be a parabolic operator with constant coefficients, with eigenvalues of a^{ij} between positive constants $\Lambda \geq \lambda$. Let $U_R^{m,1}(0) = B_R^m(0) \times (-R^2, 0]$, then*

$$||u|_{U_R^{m,1}(0)}||_{2,\alpha} \leq N \left(||Qu|_{U_{2R}^{m,1}(0)}||_{0,\alpha} + \sup_{U_{2R}^{m,1}(0)} |u| \right)$$

for any $u \in C^{2,\alpha}(U_{3R}^{m,1}(0))$, where N is some constant depending λ, Λ, α, R and m.

We also have the standard differentiability theorems from chapter 3 of [7]. First note that an operator Q is called *strictly parabolic* if $a^{ij}\zeta_i\zeta_j \geq \lambda|\zeta|^2$, for all $\zeta \in \mathbb{R}^m$, for some positive constant λ.

Theorem B.3.3. *Let Q be a strictly parabolic linear operator on $\Omega \times (S,T]$ in $\mathbb{R}^{m,1}$, with coefficients locally $C^{k,\alpha}$ on $\Omega \times (S,T]$. Then any solution of $Qu = 0$ on $\Omega \times (S,T]$ will be locally $C^{k+2,\alpha}$.*

The following is an existence theorem for quasilinear parabolic equations. We will not use it here, but it gives a nice example of the idea of short time existence, and is roughly what we would use to prove the existence of short time solutions to mean curvature flow problems.

Theorem B.3.4. *Let Ω be a smooth and bounded domain in $\mathbb{R}^m$, and let $\phi : \bar{\Omega} \times [0, T] \to \mathbb{R}$ be smooth. Let Q be a strictly parabolic quasilinear operator with coefficients smooth in all arguments. Then there exists $\epsilon > 0$ such that there exists a locally $C^{2,\alpha}$ solution to $Qu = 0$ in $\Omega \times (0, \epsilon)$ with $u = \phi$ on $\Omega \times \{0\} \cup \partial\Omega \times [0, \epsilon]$.*

This is proved as in Theorem 8.2 of [18] for single equations, but (as in section 4 of [23] for example) the same ideas can be used to get short time existence for systems. The proof again involves Schauder fixed point theorem.

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